Numerical Analysis and Curvature on Discrete Spaces

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Abstract

Multiple problems at the intersection of numerical analysis and differential geometry are explored, especially centered around the idea of understanding smooth manifolds with special types of discontinuous metrics as differential-geometric structures in their own right. The frame bundle formalism is used to derive curvature measures of such manifolds from first principles. Several aspects of the theory of finite element methods for elliptic partial differential equations on manifolds are explored in depth, with an application that shows how curvature approximations can also be used in the two-dimensional case to compute more fine-grained geometric structure numerically. Extrinsic curvatures appear in the approximation of surface tension forces acting on a fluid, and the numerical approximation of the relevant geometric quantities in multi-material finite volume codes is investigated.

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E hele kāua!

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Chapter 1

Introduction

At its most basic level, differential geometry is the study of local properties of smooth spaces which are invariant under some group of transformations. For example, one may consider spaces which are embedded surfaces in $\mathbb{R}^3$, and look for properties of such surfaces that are invariant under the group of rigid motions $x \mapsto x_0 + Ax$ for $A \in SO(3)$ and $x_0 \in \mathbb{R}^3$. This is exactly the classical theory of surfaces. One of the most important geometric properties that a smooth surface S has is whether or not it is *flat*, that is whether or not it lies inside a single plane. This is a local property in the sense that it can be verified for an arbitrarily small neighborhood, and if it is true for each neighborhood then it is true overall. Classically, there exist functions H and K, defined on S, such that S is flat if and only if $H = K = 0$ at every point. These are the mean curvature and the Gaussian curvature, respectively. Broadly, H measures average bending while K measures internal deformation.

The well-named Theorema Egregium states that K is invariant under the much larger set of *isometries* of S. An isometry of S is a map $S \to \mathbb{R}^3$ that preserves the length of every curve in S. Importantly, any data preserved by an isometry is necessarily *intrinsic*: it depends only on data internal to the surface itself, and not how that surface is embedded in $\mathbb{R}^3$. The data that an isometry preserves is called the Riemannian metric, g. This is an inner product on each tangent space of S that varies smoothly, so that angles and lengths of tangent vectors (and hence curves) can be measured.

If $K = 0$ on a simply connected neighborhood $U \subset S$, then there exists a set of coordinates $U \to \mathbb{R}^2$ such that, in these coordinates, g is equal to the usual Euclidean inner product on $\mathbb{R}^2$. So, for an observer that lives "inside" S, U is indistinguishable from Euclidean space itself. If this is true for every neighborhood $U \subseteq S$, then we say that the two-dimensional Riemannian manifold (S, g) is (intrinsically) flat. The celebrated theorem of Riemann states that, for any Riemannian manifold (M, g) (not just of dimension 2), there exists an easily computable tensor field R defined on M such that M is flat in a simply connected neighborhood if and only if $R = 0$ on that neighborhood.

However, in real life and in computer simulation, it is very rare that we can have an exact representation of a smooth manifold, or even an algebraic variety, that we would like to apply these concepts to. For example, let M be the surface of a geodesic dome. This space is clearly not flat; it should be approximately "round" in the same sense that a smooth dome is round, even though it is made up of flat pieces. This is not a Riemannian manifold in the classical sense, although it has much more structure than a general pseudomanifold. One way to approach this issue lies in Alexandrov's theorem on polyhedra [1], which roughly states that a polyhedral surface is intrinsically flat in a neighborhood of an interior vertex if and only if the sum of the interior angles of all the facets meeting that vertex is equal to 2π. More generally, for each vertex p we can define the *angle defect*,

$$\Theta_p := 2\pi - \sum_{T \ni p} \theta_T(p),$$

where $\theta_T(p)$ is the interior angle of the triangle T at the vertex p. We can then define a Gauss curvature measure $K_{\text{dist}} := \sum_{p \in M} \Theta_p \delta_p$, where δ_p is the Dirac delta measure at p. This is an intrinsic measure of curvature in the same way that the smooth Gauss curvature is; in fact, as the mesh size approaches zero, it converges to the smooth Gauss curvature measure $K dA$ [14, 28].

Figure 1.1: An example of a geodesic dome which should have Gaussian curvature approximately equal to the constant function $0.0004m^{-2}$ and mean curvature approximately equal to the constant function $0.02m^{-1}$.

Alexandrov's Theorem and its generalizations are valid for a class of pseudomanifolds called *Regge manifolds*, or alternatively, *manifolds with Regge metrics*. Intuitively, a Regge metric g is the type of metric which would be induced on M by a continuous piecewise-smooth embedding of a smooth manifold M in to another smooth manifold N, where the pieces decompose M into a simplicial (or more generally polyhedral) complex. This is a marked difference from smooth Riemannian metrics, because as an internal observer follows a straight line that crosses a polytope boundary, their velocity vector in M must be discontinuous, but only in its component that is normal to the boundary. This reflects the fact that the induced Riemannian metric g is discontinuous, but only in its normal components. It is this property that we extract to obtain the intrinsic definition of Regge metrics.

A major portion of this dissertation is devoted to deriving the intrinsic curvature measure of Regge metrics from first principles. It is already known that the higher-order Gaussian curvature measure on surfaces includes a delta distribution supported on the edges which is weighted by the jump in geodesic curvature of the edge, and a piecewise-continuous measure that is simply the usual curvature function evaluated on polytope interiors. For manifolds of dimension greater than 2, the Riemannian curvature is tensor-valued and the curvature measure must be a functional which takes tensor fields as input, but it has largely similar structure. The choice of this curvature functional is justified in [32] mainly by heuristic arguments and the fact that it converges numerically. The first chapter of this dissertation is devoted to deriving an expression for the distributional curvature from first principles that is simple to evaluate and equivalent to known expressions. This is done by defining, and proving the existence of, a class of discontinuous orthonormal frames for which a distributionally defined curvature functional satisfies basic tensoriality properties.

Besides simply deriving an expression for the distributional curvature functional, another major question that I attempt to answer is: when solving a metric-dependent PDE on a manifold with an approximate Regge metric, under what conditions can a finite element method converge as the metric converges? How should rates of convergence and mesh quality be quantified? Because operators such as the Hodge Laplacian (the covariant generalization of the Laplacian) are intrinsic, finite element methods should also converge in an intrinsic sense, meaning the rate of convergence should not depend on special coordinates chosen on the manifold. The second chapter is devoted to building the machinery of Finite Element Exterior Calculus (FEEC) on manifolds equipped with a Regge metric, including new coordinate-free proofs of the scaled trace and inverse inequalities and new coordinate-free definitions of mesh regularity.

These two lines of inquiry are united in a single numerical problem. Suppose M is a compact surface with boundary and trivial homology (so, a topological disc), and let g be a smooth metric on M. There exists an orthonormal frame on M, which is unique up to constant rotation, and such that the corresponding connection form α is co-closed. The connection form associated to this frame can be computed only knowing the values of the frame on the boundary of M and the geometry of M. Using the Hodge decomposition, it can be shown that this P.D.E. has a unique smooth solution up to a constant rotation, and it can be rewritten as a Hodge-Laplace problem on M with source and boundary terms depending on the curvature. We study this problem in detail and prove a priori error estimates using the intrinsic machinery developed earlier.

Regge metrics are mainly useful in numerical methods because one can pick finite-dimensional subspaces that approximate a smooth metric. Regge's original construction consisted of piecewise-flat Regge metrics on simplicial complexes, which are an E-dimensional subset of the space of Regge metrics, where E is the number of edges in the simplicial complex (as a specification of edge lengths determines a unique piecewise-flat Regge metric). Such metrics can be thought of as being approximate solutions to a partial differential equation involving a Riemannian metric as an unknown, such as the Ricci flow, Calabi flow, and other geometric flows. The fact that Regge metrics have a consistent and intrinsic curvature tensor is relevant for numerical methods that approximate solutions to these equations, and a thorough understanding of the curvature of Regge metrics will be necessary both to design such numerical methods and to prove their a priori convergence properties.

Regge metrics appear to be ideally suited for approximating the intrinsic Riemannian curvatures of piecewise-flat, and more generally piecewise-smooth, hypersurfaces. However, extrinsic curvatures are also relevant to physical applications. One important application is in the theory of surface tension and wetting, where mean curvature determines the magnitude of the surface tension force experienced on the interface between a self-attracting liquid and a gas or solid. This makes it very relevant for numerical simulations involving droplets, as surface tension has a strong influence on the shape of small droplets. In simplified cases, a fluid-gas interface can be tracked with an explicit triangular mesh whose nodes are directly moved, and this piecewise-flat surface has a genuine surface tension flow associated to it (see Section 4.2 for a proof). However, in many finite volume multi-material hydrocodes such as Pacific Islands Structured Arbitrary Lagrangian-Eulerian (PISALE), it is not possible or even desirable to maintain an accurate mesh representation of the interface, especially because such a mesh can easily tangle. In this case, the interface must be reconstructed during each time step as the level set of an indicator function, and it may fail to even be a continuous surface. Some related work is presented in Chapter 4 on the implementation of such a numerical scheme in PISALE.

1.1 Outline

This dissertation is split into three chapters, roughly grouped by subject matter and publication order. Each chapter is split into sections. Chapter 2 develops the theory of the distributional Riemann curvature tensor from geometric first principles, deriving a new expression which takes genuinely continuous differential forms as input and providing a proof of equivalence to existing literature. In Section 2.1, an introduction to the ideas used in the chapter and some background are given, and a rapid overview of Riemannian geometry in the framework of moving frames is included. Section 2.2.1 is a discussion of the definition of compatible frames, which I argue are the natural generalization of smooth orthonormal frames to the Regge metric case. The properties of compatible frames are used systematically to derive two curvature functionals in Section 2.2, one which is explicitly frame-dependent (2.16) and one which is frame-independent (2.22). Then, using the technical machinery of blow-ups, a constructive existence proof for compatible frames is given, making the results of this chapter non-vacuous. Lastly in Section (2.4) the Gauss-Bonnet theorem is proved for two-dimensional Regge manifolds with corners.

Chapter 3 is about specializing the framework of Geometric Variational Crimes [36] to prove intrinsic a priori convergence results for the Hodge-Laplace problem on Regge manifolds, and using this machinery to prove a priori convergence of a numerical scheme to approximate a co-closed connection form. Section 3.2 provides some background on Sobolev spaces on Regge manifolds and new intrinsic proofs of core theorems in the theory of finite element methods, namely the scaled trace inequality and the scaled inverse inequality. These are stated with no reference to any special coordinates, meaning they can be applied fruitfully in any coordinate chart without worry about poor bounds due to poor choice of coordinates (which cannot always be avoided). This leads naturally to new definitions of shape-regularity and quasi-uniformity for meshes with Regge metrics, which are discussed in detail. Section 3.3 specializes the framework of geometric variational crimes to the Regge metric case, and Sections 3.4-3.5 pose and then solve the problem of computing a uniquely defined connection form based on all the previous results, with a short numerical demonstration.

Chapter 4 is about approximating surface tension flow in the PISALE application. After giving a geometrical derivation of the surface tension force in terms of surface energy in Section 4.2, I describe PISALE and the Continuum Surface Force (CSF) method that is used in PISALE's surface tension module.

The remainder of this introduction is devoted to introducing notation and objects that are used frequently throughout the document.

1.2 Differential Forms and Exterior Calculus

Here it is useful point out two pieces of notational technology that are used comprehensively throughout this document. The first is the use of differential forms and exterior calculus. Before giving precise definitions, some intuition will be provided.

Intuitively, a differential k-form can be understood as "something that can be integrated over an oriented k-dimensional manifold". For example, functions are differential 0-forms, which measure the "volume" of a single point by evaluation. The integrand of the Riemann integral $\int_a^b f\ dx$ can be thought of as the differential 1-form $f\ dx$ integrated over the 1-dimensional manifold $[a, b]$, with the orientation induced by the ordering. As with Riemann integration, smooth differential forms satisfy a change of variables formula: $\int_{f(M)} \alpha = \int_M f^*\alpha$, where $f : M \to N$ is a smooth embedding and $f^*\alpha$ is the *pullback* of α by f, which generalizes taking the determinant of the Jacobian map of f.

In a more complex example, consider the surface integral $\int_S V \cdot \vec{n}_S dS$, where $S \subset \mathbb{R}^3$ is a C^1 surface with outward-pointing normal vector $\vec{n}_S$ and V is a vector field defined on $\mathbb{R}^3$. The form $V \cdot \vec{n}_S dS$ is a 2-form defined only on S, but it is actually identical to the form $i_S^*\alpha$, where $\alpha(X, Y) = V \cdot (X \times Y)$. α itself is defined independently of S, and $i_S^*\alpha$ is the pullback by the inclusion map $i_S : S \hookrightarrow \mathbb{R}^3$, essentially restricting α to S. The fact that $V \cdot \vec{n} dS$ actually corresponds to an object that exists on all of $\mathbb{R}^3$ helps greatly in gaining insight to the divergence theorem.

Formally, a k-form α defined on a manifold M is the data, at each point $x \in M$, of an alternating multilinear map $T_xM \times \cdots \times T_xM \to \mathbb{R}$, where T_xM is the tangent space of M at x. Here "alternating" means that when any adjacent pair V_i, V_{i+1} of input vectors are swapped, the output changes sign. An alternative way to phrase this is that α is a section of the alternating k-covector bundle, defined as $\Lambda^k(M) := \{(x, F) : x \in M, F \in \mathrm{Alt}(\bigotimes^k T_x^*M)\}$. Usually, but not always, we will require that α is at least continuous. The value $\alpha|_x(V_1,\dots,V_k)$ can be thought of intuitively as the oriented k-dimensional volume, as defined by $\alpha|_x$, of the infinitesimal parallelepiped spanned by the vectors $V_1,\dots,V_k$ in T_xM. A differential form α is a differential k-form for some value of k; the *degree* $\deg(\alpha)$ is equal to k. The space of smooth differential k-forms on M is a $C^\infty(M)$-module denoted by $C^\infty\Omega^k(M)$. The pullback of $\alpha \in C^\infty\Omega^k(N)$ by a map $f : M \to N$ is formally defined by $f^*\alpha(V_1,\dots,V_k) = \alpha(df(V_1),\dots,df(V_k))$, where df is the Jacobian map of f.

The wedge product of a k-form α and a j-form β is a $(k+j)$-form $\alpha\wedge\beta$, defined by

$$(\alpha\wedge\beta)(V_1,\dots,V_{k+j}) := \frac{1}{k!j!}\sum_{\sigma\in S_{k+j}} \operatorname{sgn}(\sigma)\alpha(V_{\sigma(1)},\dots,V_{\sigma(k)})\beta(V_{\sigma(k+1)},\dots,V_{\sigma(k+j)}),$$

where S_{k+j} is the set of permutations of the set $\{1,2,\dots,k+j\}$, and $\operatorname{sgn}(\sigma)$ is 1 if the permutation is even (meaning it decomposes into an even number of transpositions) and -1 if it is odd. This product is chosen so that it is associative and satisfies $\beta\wedge\alpha = (-1)^{\deg(\alpha)\deg(\beta)}\alpha\wedge\beta$. Additionally, if $\alpha^1,\dots,\alpha^k$ are one-forms, then

$$(\alpha^1\wedge\cdots\wedge\alpha^k)(X_1,\dots,X_k) = \sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\prod_{i=1}^k \alpha^i(X_{\sigma(i)}) = \det\left(\left[\alpha^i(X_j)\right]\right).$$

It is often helpful to think of differential forms as taking values in a vector space, or even in a vector bundle. If W is a vector space or a vector bundle, $C^\infty\Omega^k(M;W)$ will be used to denote the space $W\otimes_{\mathbb{R}} C^\infty\Omega^k(M)$ if W is a vector space or $\Gamma(W)\otimes_{C^\infty(M)} C^\infty\Omega^k(M)$ if W is a vector bundle over M, with $\Gamma(W)$ denoting the space of sections of W. Operations such as the pullback and wedge product implicitly operate on the 'form part' of a vector-valued or bundle-valued form (for vector-valued forms, the exterior derivative is also defined in this way).

By picking a smooth coordinate system $(x^1,\dots,x^n)$ in a neighborhood, we get a basis of 1-forms $(dx^1,\dots,dx^n)$ defined by $dx^i(e_j) = \delta^i_j$, where δ is the Kronecker symbol. Any smooth differential k-form α can be written as

$$\sum_{1\le i_1<i_2<\cdots<i_k\le n} \alpha_{i_1i_2\dots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k},$$

Where each coefficient $\alpha_{i_1 i_2 \dots i_k} = \alpha(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}})$ is a smooth function.

The summation signs and indexing above can become unwieldy in high-dimensional tensor calculus, motivating a second piece of notational technology. A *multi-index* is a strictly increasing sequence of numbers $I = (i_1, \dots, i_k)$, where $1 \leq i_1$ and $i_k \leq n$. The order of a multi-index is its length, and is written $|I|$. If $(\alpha^1, \dots, \alpha^n)$ is a basis of one-forms, then we define $\alpha^I := \alpha^{i_1} \wedge \dots \wedge \alpha^{i_k}$. The forms $\{\alpha^I|_x\}_{|I|=k}$ form a basis for $\Lambda^k_x(M)$ at each $x \in M$.

In Einstein convention, elements of T_xM and tensor products thereof (*contravariant* tensors) are enumerated with lower indices, while elements of the corresponding dual space (*covariant* tensors) are enumerated with upper indices. The coefficients of a contravariant tensor expressed in some basis have upper indices, while the coefficients of a covariant tensor expressed in some basis have lower indices. This reflects how the coefficients change when a change of basis is applied; for instance, when changing from the covariant basis $(e_1, \dots, e_n)$ to $(\hat{e}_1, \dots, \hat{e}_n) = (\sum_j M^j_1 e_j, \dots, \sum_j M^j_n e_j)$, the coefficients of a vector $V = \sum_k V^k e_k$ change to $\hat{V}^k = \sum_j V^j (M^{-1})^k_j$, so that $\sum_k V^k e_k = V = \sum_k \hat{V}^k \hat{e}_k$.

A profound insight of Einstein is that, for any operation that does not depend on a metric, any pairing must be between contravariant and covariant parts of tensors, which will always correspond to pairing an upper index with a lower index. So, when writing the summations above, we can simply drop the summation symbols, and say that any instance of a lower index matching an upper index will mean there is an implicit summation. Any other type of summation must be explicit. In the case of multi-indices, the degree of the multi-indices to sum over must be implicitly known.

So, when α is a differential k-form, we will simply write its coordinate expansion as

$$\alpha = \alpha_I dx^I.$$

In $\mathbb{R}^3$, the exterior derivative unifies the gradient, curl, and divergence operators, at the same time giving them coordinate-independent meaning. Given a C^1 differential k-form, it produces a differential $(k+1)$-form according to the following definition: if f is a function and α, β are forms of arbitrary degree, then

$$df := \frac{\partial f}{\partial x^i} dx^i,$$
$$d(\alpha \wedge \beta) := d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta.$$

There are three properties of the exterior derivative that are used comprehensively, but will not be proved here (see [20, 25]). Firstly, it is independent of coordinates, and thus represents a genuine operation on differential forms themselves. Secondly, $d(d\alpha) = 0$ for any twice differentiable differential form α, essentially because mixed partial derivatives commute. Lastly, a vast generalization of the usual integration by parts theorems in vector calculus can be proved for differential forms:

$$\int_M d\alpha = \int_{\partial M} i^*_{\partial M} \alpha.$$

Here M is a n-dimensional manifold with (smooth) boundary denoted by ∂M, and α is a smooth differential $(n-1)$-form defined on M. In fact, a generalization for nonsmooth forms and manifolds with more complicated boundaries is used throughout this dissertation; this is explained in detail in the next section.

1.3 Polyhedral Manifolds, Meshes, and Regge Metrics

In classical differential geometry, the metrics and frames in question are always smooth. However, this paper is concerned with Regge metrics, which are only piecewise smooth with respect to a mesh. In this section we will precisely define much of the terminology of meshes and Regge metrics that is used throughout the rest of the paper.

M will be assumed to be a smooth, oriented, polyhedral n-manifold (the precise definition of a polyhedral manifold comes later in this section). We will also assume that M is parallelizable, with the idea in mind that more topologically complicated manifolds can be obtained by gluing together finitely many parallelizable ones; for instance a sphere can be obtained by gluing together two disks. M is equipped with a countable *mesh* $\mathcal{T} = \{\Delta\}$ whose union is M, where each set $\Delta \subset M$ is the image of a closed $(n-d)$-dimensional convex polytope $\hat{\Delta} \subset \mathbb{R}^{n-d}$ under a smooth embedding for some $d \leq n$. If $d = 0$, then the embedding must in addition be positively oriented. We will abuse terminology and call each Δ a polytope, even though it is technically the image of a polytope under a smooth embedding.

The different types of polytope are distinguished by their codimension in M; polytopes of codimension 0 will be labeled T, polytopes of codimension 1 will be labeled e, and polytopes of codimension 2 will be labeled p. Polytopes of arbitrary codimension will simply be labeled Δ_d, where d is the codimension, so a polytope labeled Δ_n is a single point, a polytope labeled Δ_{n-1} is a smooth curve, and so on. The relative interior of a polytope Δ_d, meaning the set of all points $x \in \Delta_d$ such that there exists an open set U containing x and $U \cap \Delta_d \cong \mathbb{R}^{n-d}$, will be denoted $\mathring{\Delta}_d$.

The mesh must satisfy some axioms. Each face of a polytope in the mesh is also a polytope of the mesh. The intersection of two polytopes must be either a shared face of both polytopes or empty. Additionally, the mesh must respect the stratification structure of M. Being a polyhedral manifold, the boundary ∂M can be decomposed into strata $S_d(M)$, which consists of those points $x \in \partial M$ which are contained in a submanifold of ∂M that is of codimension d, but not one that is of codimension $d-1$. We will require that if the relative interior of a polytope Δ_d intersects $S_{d'}(M)$, then $\mathring{\Delta}_d \subseteq S_{d'}(M)$. This prevents pathological tangencies at the boundary, and it means that the closure of each stratum of M inherits its own mesh decomposition.

A *polyhedral n-dimensional manifold* M is a smooth manifold which is locally modeled on relatively open subsets of nondegenerate unions of parallelepipeds. This class of manifolds includes all polytopes and all domains which can be obtained by identifying faces of convex polytopes in $\mathbb{R}^n$ by rigid motions.

Specifically, every point $x \in M$ has a coordinate neighborhood (U, ϕ_U) where ϕ_U is a one-to-one open map $U \to R_U \subset \mathbb{R}^n$. $R_U = P_1 \cup \cdots \cup P_N$, where each set P_i is a closed nondegenerate n-dimensional parallelepiped in $\mathbb{R}^n$ and the intersection $P_i \cap P_j$ is either empty or a shared face of both P_i and P_j. A continuous map $f : A \subseteq R_U \to R_{U'}$, where A is a relatively open subset of R_U, is considered a smooth map if it can be extended to a smooth map $f : \mathbb{R}^n \to \mathbb{R}^n$. The smooth structure is given by a maximal atlas of coordinate patches (U, ϕ_U) such that the transition maps $\phi_U \circ \phi_{U'}^{-1}$ are all smooth with this definition. As usual, M is also required to be a second-countable Hausdorff space.

Note that if $\psi : A \subseteq R_U \to B \subseteq R_{U'}$ is a diffeomorphism, then each face $\Delta_d \subset \partial R_U$ must map to a face Δ'_d of $\partial R_{U'}$ of the same codimension. For all $0 < d \leq n$, let $S_d(R_U)$ be defined as the union of relative interiors of faces of ∂R_U which have codimension d, and define $S_0(R_U) = \mathring{R}_U$. So, if $\phi_U(x) \in S_d(R_U)$, then $\phi_{U'}(x) \in S_d(R_{U'})$ for all smooth charts $U' \ni x$. The set of such points in M therefore defines a union of disjoint submanifolds (without boundary) $S_d(M) := \bigcup_U \phi_U^{-1}(S_d(R_U)) = \bigcup_U \bigcup_{\Delta_d \subset S_d(R_U)} \phi_U^{-1}(\mathring{\Delta}_d)$, called the d-stratum of M. The closure of a connected component of $S_d(M)$ is also a polyhedral manifold, and it is called a codimension-d face of M, so called because it lies in a codimension-d face in each coordinate chart U. Since $\bigcup_{d\geq 1} S_d(R_U) = \partial R_U$ for each coordinate chart U, we can also define $\partial M := \bigcup_{d\geq 1} S_d(M)$ and $E_M := \bigcup_{d\geq 2} S_d(M)$. The set E_M is closed and has $(n-1)$-dimensional Minkowski content zero in any coordinate chart, and it will be called the exceptional set.

A version of Stokes' theorem can be produced for polyhedral manifolds, based on the fact that they satisfy all the axioms of Whitney manifolds: if ω is in $C^1\Omega^{n-1}(\mathring{M})$ and bounded on $M \backslash E_M$ (meaning each coefficient is bounded in any coordinate chart), $\omega|_{\partial M \backslash E_M}$ is summable, and $d\omega|_{\mathring{M}}$ is summable, then [61, p. 108]

$$\int_{\mathring{M}} d\omega = \int_{\partial M \backslash E_M} \omega = \int_{S_1(M)} \omega.$$

Sometimes it will be necessary to use the more restrictive class of *manifolds with corners.* Using notation from above, this would be the class of polyhedral manifolds such that each of the coordinate charts U map into a set R_U that consists of a single solid n-dimensional cube. Manifolds with corners thus have substantially less rigidity (fewer smoothly distinct neighborhoods) than general polyhedral manifolds, which is relevant for some boundary integrals.

Chapter 2

Discrete Riemann Curvature

2.1 Introduction

Since Tullio Regge's introduction of what are today called Regge metrics—discontinuous metrics with tangential-tangential continuity—they have found widespread use in numerical models of general relativity [42, 52], continuum mechanics [42, 49, 50], and more. In many of these applications, one of the key features of a Regge metric is that curvature measures can be defined which converge in measure to their smooth counterparts. Regge's original paper [52] discussed piecewise-constant metrics on simplicial meshes, where the scalar curvature is given simply by the angle defect around codimension-2 interfaces. Further investigations by Cheeger, Müller, and Schrader [14] proved that a broader class of curvatures, called the Lipschitz-Killing curvatures, converges in measure to their smooth counterparts in the piecewise-flat setting.

Later developments involved proving convergence results for Gauss, scalar, and Einstein curvature measures on higher-order Regge metrics, where the metric is piecewise-smooth rather than only piecewise-flat but retains the jump conditions on simplex boundaries [6, 27–29]. The angle defect remains part of all of these curvature measures, but extra terms involving the curvature on element interiors and the jump in mean curvature/second fundamental form over element boundaries also appear. Only recently have convergence results been proved for a distributional version of the full Riemann curvature tensor in arbitrary dimension [32].

In this chapter we will focus not on the numerical convergence of distributional curvature, but rather on deriving the "correct" definition of distributional curvature from first principles. The formulas (2.16) and (2.22) are equivalent to that for the densitized distributional Riemann curvature described in [32]. We are not the first to pursue a derivation of the correct definition from first principles, especially for the scalar curvature in the piecewise-flat setting; see [8, 9, 15, 16] and the references therein for other perspectives on this.

Our starting point is the structure equations on the orthonormal frame bundle, about which some background is provided in the following section. We take them to be the "ground truth" that define the distributional curvature. The question is, then, which moving frames f can be used to obtain a distributional curvature functional $f^*\Omega_{\text{dist}}$ that is actually a measure (or more precisely an order-0 current), and that has the correct gauge transformation law? We argue that the vectors comprising such a frame f need to possess single-valued normal and tangential components on codimension-1 interfaces, forcing them to be discontinuous at (generic) codimension-2 interfaces. A frame f with this property will be called "compatible" if it satisfies a few regularity hypotheses and topological constraints detailed in Definition 1. We argue that compatible frames are the correct generalization of smooth orthonormal frames. Section 2.2.1 is a discussion motivating the definition of compatible frames. Sections 2.2.2-2.2.3 then use this definition to derive an expression for the distributional curvature functional $f^*\Omega_{\text{dist}}$.

Importantly, a core part of the definition of a compatible frame involves *blow-ups* of polytopes, essentially to ensure that the frame is regular enough to permit integration by parts, even though it is discontinuous. The idea of using blow-ups to define geometric invariants of Regge metrics is not new, appearing in [5, p. 2] for much the same reason why we use it here. In Section 2.2, after deriving properties that such a frame must have, we use them to obtain an expression for $f^*\Omega_{\text{dist}}$ that behaves as it should. We use blow-ups to help compute $f^*\Omega_{\text{dist}}$ in Sections 2.2.2-2.2.3, and we also use them in Section 2.2.5 to derive a frame-independent expression for the distributional curvature, which is more practical for real computations, especially on manifolds that are not necessarily parallelizable.

Our main results are stated in Theorems 1 and 2. Roughly speaking, they say that if an n-dimensional manifold M is equipped with a (curvilinear) polyhedral mesh, a Regge metric g, and a compatible frame f, then the structure equations for the connection one-form and curvature two-form can be given meaning in a distributional sense. Furthermore, the distributional curvature two-form $f^*\Omega_{\text{dist}}$, when reinterpreted as an $\text{End}(TM)$-valued two-form, is a functional $\hat{R}_{\text{dist}}$ that acts on any $\text{End}(TM)$-valued $(n-2)$-form $\hat{\phi}$ with sufficient regularity via

$$\langle\langle \hat{R}_{\text{dist}}, \hat{\phi} \rangle\rangle = \sum_{T \subseteq M} \int_{\mathring{T}} \langle \hat{R} \wedge \hat{\phi} \rangle - \sum_{\mathring{e} \subset \mathring{M}} \int_{\mathring{e}} \left[\!\left[\langle \hat{\mathrm{I\!I}}_e \wedge i_{\mathring{e}}^* \hat{\phi} \rangle \right]\!\right] + \sum_{\mathring{p} \subset \mathring{M}} \int_{\mathring{p}} \langle \hat{\Theta}_p \wedge i_{\mathring{p}}^* \hat{\phi} \rangle.$$

Here, the sums over T, e, and p are sums over polytopes of codimension 0, 1, and 2, respectively, and the maps $i_{\mathring{e}}^*$ and $i_{\mathring{p}}^*$ are pullbacks under the inclusions $\mathring{e} \hookrightarrow M$ and $\mathring{p} \hookrightarrow M$. The notation $[\![\cdot]\!]$ specifies the jump of a multi-valued quantity over the submanifold $e = T \cap T'$, which in this case is simply the difference $\langle \hat{\mathrm{I\!I}}_e^T \wedge i_{\mathring{e}}^* \hat{\phi}|_T \rangle - \langle \hat{\mathrm{I\!I}}_e^{T'} \wedge i_{\mathring{e}}^* \hat{\phi}|_{T'} \rangle$. The quantities $\hat{R}$, $\hat{\mathrm{I\!I}}_e^T$, and $\hat{\Theta}_p$ are $\text{End}(TM)$-valued 2-forms, 1-forms, and 0-forms that encode the curvature tensor, second fundamental form, and angle defect, respectively. The operation $\langle \cdot \wedge \cdot \rangle$ takes a pair of $\text{End}(TM)$-valued forms and wedges their form parts and applies a nondegenerate pairing to their endomorphism parts.

Constructing compatible frames is nontrivial, and we provide an existence proof in Section 2.3. Lastly, in Section 2.4 we investigate the specialization to two dimensions, where the Gauss curvature measure can be defined independently of the frame, and we prove that there is a suitable generalization of the Gauss-Bonnet theorem.

Our results are stated for extremely general oriented parallelizable manifolds and meshes.

This is partly because the presentation is actually not much more complicated for a general polyhedral mesh, since many delicate analytical conditions for Stokes' theorem must be dealt with even in the simplest cases, and existing literature treating this subject has already been developed to a very high level of generality. One caveat is that we rely on the existence of blow-ups of polytopes which are also polytopes, but we have only been able to locate an existence theorem for blow-ups of simplices in the literature. An explanation of the types of manifolds we use in this paper, and references to relevant literature, can be found in the appendix.

Some opportunities for extensions of our results are immediately apparent. The method of moving frames generalizes quite simply to indefinite (also called pseudo-Riemannian) metrics and to more general geometries as well. Lemma 4 in Section 2.3 is already stated for arbitrary pseudo-Riemannian metrics. While the conditions we set for compatible frames make key use of the particular properties of the structure group $O(n)$, it is clear where the dependence lies and what would constitute removing it. Essentially the difficulty will lie in the jump conditions at codimension-2 polytopes, and correspondingly in the angle defect terms of the distributional curvature equation. These must be phrased in terms of another one-parameter group adapted to the geometry. We use the integration theory of differential forms whenever possible, avoiding metric dependence.

2.1.1 Background: Geometry in a Moving Frame

In the method of moving frames [37], one considers the geometry of a manifold by finding general constructions in the frame bundle, and then choosing an adapted frame which is most convenient for calculations.

Consider an oriented, parallelizable polyhedral n-manifold M furnished with a smooth Riemannian metric g. The *frame bundle* of M, denoted $\mathcal{F}_{GL}(M)$, is the sub-bundle of $TM \times \cdots \times TM = (TM)^n$ such that the fiber over each point $x \in M$ is the set of ordered bases $F = (F_1, \ldots, F_n)$ for T_xM. The *orthonormal frame bundle* of M, denoted $\mathcal{F}_O(M)$, is the sub-bundle of $\mathcal{F}_{GL}(M)$ such that the fiber over each point is the set of ordered bases satisfying $\langle F_i, F_j \rangle = \delta_{ij}$ for all i, j. A *frame* is a section of the frame bundle, and an *orthonormal frame* is a section of the orthonormal frame bundle.

There is a right action of $GL(n)$ on each fiber of $\mathcal{F}_{GL}(M)$ defined by $(F \cdot h)_i = F_j h^j_i$. (To be consistent with Einstein summation notation, the entry of a matrix at the jth row and ith column will be written h^j_i). Clearly if $(x, F) \in \mathcal{F}_O(M)$ and $h \in O(n)$, then $(x, F \cdot h) \in \mathcal{F}_O(M)$ as well. On Riemannian manifolds, orthonormal frames are often more convenient to work with than coordinate frames.

The frame bundle has a canonical construction called the *solder form*. This is a vector-valued one-form $\theta \in C^\infty\Omega^1(\mathcal{F}_{GL}(M); \mathbb{R}^n)$ defined implicitly by

$$d\pi \left(v|_{(x,F)} \right) = \theta^j \left(v|_{(x,F)} \right) F_j,$$

where $\pi : \mathcal{F}_{GL}(M) \to M$ is the bundle projection and $d\pi : T\mathcal{F}_{GL}(M) \to TM$ is its tangent map. When f is a smooth frame, the one-forms $\{f^*\theta^j\}_{j=1}^n$ form a basis of the cotangent space $T^*_x M$ such that $(f^*\theta^j)(f_k) = \delta^j_k$.

A vector $v|_{(x,F)} \in T_{(x,F)}\mathcal{F}_{GL}(M)$ will be called *vertical* if $\theta(v) = 0$. Since the group action of $GL(n)$ on the fibers of $\mathcal{F}_{GL}(M)$ is free, all vertical vectors are derivatives at time $t = 0$ of curves $t \mapsto (x, F \cdot h_v(t))$, where $h_v : (-1, 1) \to GL(n)$ is a smooth curve with $h_v(0) = I$. This means we can define a linear map $\eta : \ker \theta \subset T_{(x,F)}\mathcal{F}_{GL}(M) \to \mathfrak{gl}(n)$ by $v \mapsto \dot{h}_v(0)$. This is a linear isomorphism at each point, with inverse $\eta^{-1}(\dot{h}(0)) = \frac{\partial}{\partial t}|_{t=0}(x, F \cdot h(t))$. When restricted to vertical vectors in $T_{(x,F)}\mathcal{F}_O(M)$, η becomes $\mathfrak{so}(n)$-valued.

It can be shown that η defined this way is smooth by taking a coordinate neighborhood $U \subset M$, which defines a smooth section $s : U \to \mathcal{F}_{GL}(U)$ by $s_i := \frac{\partial}{\partial x_i}$. Every point $(x, F) \in \mathcal{F}_{GL}(U)$ is then equal to $(x, s \cdot h)$ for some matrix $h \in GL(n)$, which means $\mathcal{F}_{GL}(U)$ has smooth coordinates (x^i, h^j_k). When expressed in these coordinates, η is equal to $h^{-1}dh$. Thus η is called the (left-invariant) *Maurer-Cartan form*, ubiquitous in the theory of the geometry of Lie groups and symmetric spaces [58].

One of the classical theorems in Riemannian geometry is the existence and uniqueness of a metric-compatible, torsion-free connection called the Levi-Civita connection. Similar reasoning [26] can be used to derive the existence and uniqueness of an $\mathfrak{so}(n)$-valued form $\omega \in \Omega^1(\mathcal{F}_O(M); \mathfrak{so}(n))$ such that, in $\mathcal{F}_O(M)$,

$$d\theta^i = -\omega^i_j \wedge \theta^j. \tag{2.1}$$

This form ω encodes the Levi-Civita connection. In fact, if ∇ is the usual Levi-Civita connection and $f : M \to \mathcal{F}_O(M)$ is a smooth orthonormal frame, then $f^*\omega^i_j(v) = \langle \nabla_v f_j, f_i \rangle$. Furthermore, the curvature of this connection,

$$\Omega := d\omega + \frac{1}{2}[\omega, \omega], \tag{2.2}$$

written in coordinates as $\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j$, is equivalent to the Riemann curvature tensor in the sense that $f^*\Omega^i_j(X, Y) = \langle R_{X,Y} f_j, f_i \rangle$, where $R_{X,Y} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$. More geometric identities, such as the Bianchi identities, can be derived from these *structure equations*, but these two are sufficient for the purposes of this paper.

For convenience we will introduce a basis of $\mathfrak{so}(n)$ that is used throughout this paper. For $i \neq j$, let $w^i_j \in \mathfrak{so}(n)$ be defined as the matrix such that the entry at the ith row and jth column is equal to -1, while the entry at the jth row and ith column is equal to 1, and all other entries are zero. Every element $A \in \mathfrak{so}(n)$ can therefore be written as $\sum_{i<j} A^j_i w^i_j$.

The last piece of background from Lie group theory that is needed is the *adjoint action*. It comes from the conjugation action of $GL(n)$ on itself, which we will denote $\mathbf{Ad}(h)(k) := hkh^{-1}$. The adjoint action $\mathrm{Ad} : GL(n) \to \mathrm{Aut}(\mathfrak{gl}(n))$ is defined by $\mathrm{Ad}(h)(A) = hAh^{-1}$. Abstractly, $\mathrm{Ad}(h)(A)$ is the derivative of $\mathbf{Ad}(h)(k(t))$ at $t = 0$, with k being any curve such that $k(0) = I, \dot{k}(0) = A$.

A $\mathfrak{gl}(n)$-valued form α on $\mathcal{F}_{GL}(M)$ is said to be *tensorial* (or *semi-basic*) if $v \lrcorner \alpha = 0$ for any vertical vector field v (so α evaluates to zero on any multivector that has a vertical component) and $\alpha|_{(x,F\cdot h)} = \mathrm{Ad}(h^{-1})(\alpha|_{(x,F)})$ for any $h \in O(n)$. This expresses the idea that α, in some sense, does not depend on the frame; if α is tensorial, then an endomorphism-valued form $\hat{\alpha} \in \Omega^k(M; \mathrm{End}(TM))$ could be defined so that $F_i \alpha^i_j|_{(x,F)} = \hat{\alpha}|_x(F_j)$.

The 2-form Ω defined above is tensorial, but the 1-form ω is not. In fact, if we were to calculate out $\omega(v)$ where v is a vertical vector, then we get the same as $\eta(v)$. This can also be expressed by the gauge transformation law: if $f : M \to \mathcal{F}_O(M)$ is an orthonormal frame and $h : M \to O(n)$ is a change of basis, then

$$(f \cdot h)^*\omega = h^{-1}dh + \mathrm{Ad}(h^{-1})(f^*\omega). \tag{2.3}$$

In other words, the Levi-Civita connection is not a tensor.

Remark 1. *It must be noted that, in this section, we have treated M as if there can exist a global frame $f : M \to \mathcal{F}_{GL}(M)$. In many cases, this is not possible. Manifolds that do support a global frame are called* parallelizable*. However, every point in a manifold is contained in a neighborhood which is parallelizable. Since curvature is a local property, this means the parallelizable case is really the most interesting. Parallelizability is also different from being topologically trivial; for instance, spheres of dimension* 0*,* 1*,* 3*, and* 7 *and products thereof are all parallelizable.*

We make frequent use of a "wedge inner product" between forms that take values in $\mathfrak{gl}(n)$. We can define a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the vector space $\mathfrak{gl}(n) = \mathbb{R}^{n\times n}$ by setting $\langle A, B\rangle := \mathrm{Tr}(AB)$. When restricted to $\mathfrak{so}(n)$, it is a negative-definite inner product. This means we can define a product

$$\langle \cdot \wedge \cdot \rangle : (\mathfrak{gl}(n) \otimes \Lambda^k_x(M)) \otimes (\mathfrak{gl}(n) \otimes \Lambda^{n-k}_x(M)) \to \Lambda^n_x(M)$$

by setting

$$\langle (A \otimes \alpha) \wedge (B \otimes \beta)\rangle := \langle A, B\rangle \alpha \wedge \beta$$

and extending multilinearly. This is not a true inner product, but it is nondegenerate and bilinear, and it is completely independent of any metric structure, which makes it desirable for our use case. It also has some useful symmetries. One we will use often is that $\langle(\mathrm{Ad}(h)(A)\otimes\alpha)\wedge(B\otimes\beta)\rangle = \langle(A\otimes\alpha)\wedge(\mathrm{Ad}(h^{-1})(B)\otimes\beta)\rangle$ for any $h \in GL(n)$. When applying the adjoint action (or any other map $\mathfrak{gl}(n) \to \mathfrak{gl}(n)$) to a Lie algebra part of a Lie algebra valued form, we will abuse notation slightly by applying it to the whole form.

2.2 Derivation of the Distributional Riemann Curvature

The distributional Riemann curvature tensor associated to an orthonormal frame f, which we will denote by $f^*\Omega_{\text{dist}}$, is a linear functional which associates a number to each smooth, compactly supported $\mathfrak{so}(n)$-valued $(n-2)$-form ϕ which vanishes when pulled back to ∂M. The frame f is made up of C^2 frames on the interior of each codimension-0 polytope T, so $f = \bigsqcup_{T \subseteq M} f^T$ and each $f^T : \mathring{T} \to \mathcal{F}_O(\mathring{T})$ is a C^2 section. We will also take as a definition that $f^*\omega$ and $f^*[\omega, \omega]$ are the piecewise-C^2 forms defined as $f^*\omega|_{\mathring{T}} := f^{T^*}\omega$ and $f^*[\omega, \omega]|_{\mathring{T}} := f^{T^*}[\omega, \omega]$, where $\omega \in C^1\Omega^1(\mathcal{F}_O(\mathring{T}); \mathfrak{so}(n))$ is the usual connection one-form. The distributional exterior derivative of $f^*\omega$ is the linear functional defined by

$$\langle\langle df^*\omega, \phi \rangle\rangle := \sum_{T \subseteq M} \int_{\mathring{T}} \langle f^{T^*}\omega \wedge d\phi \rangle \tag{2.4}$$

for all $\phi \in C_c^\infty\Omega^{n-2}(M; \mathfrak{so}(n))$ that vanish when pulled back to ∂M.

Per the discussion above, our definition of $f^*\Omega_{\text{dist}}$ is

$$\langle\langle f^*\Omega_{\text{dist}}, \phi \rangle\rangle := \langle\langle df^*\omega + \frac{1}{2} f^*[\omega, \omega], \phi \rangle\rangle = \sum_{T \subseteq M} \int_{\mathring{T}} \langle f^{T^*}\omega \wedge d\phi \rangle + \frac{1}{2} \langle f^{T^*}[\omega, \omega] \wedge \phi \rangle. \tag{2.5}$$

What we aim to do is find conditions on the frame f such that the right-hand side of (2.5) can be efficiently computed, is bounded by a multiple of the supremum norm of ϕ, and transforms like a tensor.

2.2.1 Conditions on Compatible Frames

An orthonormal frame $f = \bigsqcup_{T \subseteq M} f^T$ will be called *compatible* if it has some desirable properties that make $\langle\langle f^*\Omega_{\text{dist}}, \phi \rangle\rangle$ both correct from a geometrical standpoint and practical from a computational standpoint. The conditions that a compatible frame must satisfy are fairly technical and may seem arbitrary, so before stating them, we will first provide some motivation.

First, we want the individual vector fields in our frame to be "parallel" across codimension-1 polytopes, in some sense. For each interior codimension-1 polytope $\mathring{e} \subset \mathring{M}$, we can define a frame E_e which is orthonormal in the metric $i_e^* g$; this metric is well-defined since g has single-valued tangential-tangential components on e. If $e = T \cap T'$, then E_e can be extended to two orthonormal frames E_e^T and $E_e^{T'}$ by appending outward-facing normal vectors to e for g^T and $g^{T'}$ respectively, which we denote by $\vec{n}$ and $\vec{n}'$. Assume that for each T, f^T can be continuously extended to $\mathring{e}$ by the C^1 section $f^T|_{\mathring{e}} : \mathring{e} \to \mathcal{F}_O(T)|_{\mathring{e}}$, and let $\mu_e^T : \mathring{e} \to O(n)$ be a map such that

$$E_e^T \cdot \mu_e^T = f^T|_{\mathring{e}}$$

and likewise $E_e^{T'} \cdot \mu_e^{T'} = f^{T'}|_{\mathring{e}}$. The condition on codimension-1 faces is that

$$\mu_e^T = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \mu_e^{T'}.$$

That is, the tangential components of each f_i are continuous and the normal component of f_i is continuous if one of the normal vectors is negated, so $\langle f_i, \vec{n} \rangle_T = \langle f_i', -\vec{n}' \rangle_{T'}$.

While somewhat arbitrary, this notion of "parallelism" is supported by the fact that a piecewise-smooth geodesic, defined as a locally energy-minimizing curve, must have a velocity vector that satisfies the same condition we have placed on the f_i's.

Another, possibly deeper reason for this to be true, is that we need some kind of frame-independent coupling across codimension-1 polytopes for the distributional curvature to be tensorial. Suppose a piecewise-smooth frame f is compatible and can be continuously extended to each codimension-1 boundary component $\mathring{e} \subset \partial T$ for each $T \subseteq M$. Then if ϕ has support in a small neighborhood of a point $x_0 \in \mathring{e}$, where $e = T_1 \cap T_2$ is an interface between two polytopes, Stokes' theorem gives us

$$\begin{aligned}
\langle\langle f^*\Omega_{\text{dist}}, \phi\rangle\rangle &= \langle\langle df^*\omega + \frac{1}{2}f^*[\omega,\omega], \phi\rangle\rangle \\
&= \sum_{i=1,2}\int_{\mathring{T}_i}\left(\langle {f^{T_i}}^*\omega \wedge d\phi\rangle + \frac{1}{2}\langle {f^{T_i}}^*[\omega,\omega]\wedge\phi\rangle\right) \\
&= \sum_{i=1,2}\int_{\mathring{T}_i}\langle {f^{T_i}}^*(d\omega + \frac{1}{2}[\omega,\omega])\wedge\phi\rangle - \int_{\mathring{e}}\langle [\![f^*\omega]\!] \wedge \phi\rangle \\
&= \sum_{i=1,2}\int_{\mathring{T}_i}\langle {f^{T_i}}^*\Omega\wedge\phi\rangle - \int_{\mathring{e}}\langle [\![f^*\omega]\!] \wedge \phi\rangle,
\end{aligned}$$

where $[\![f^*\omega]\!]$ denotes the jump in $f^*\omega$ across e. This expression does not depend on any derivatives of ϕ, so the domain of $f^*\Omega_{\text{dist}}$ can be formally extended to include piecewise-smooth forms (with the same support) by setting

$$\langle\langle f^*\Omega_{\text{dist}}, \phi\rangle\rangle = \sum_{i=1,2}\int_{\mathring{T}_i}\langle {f^{T_i}}^*\Omega\wedge\phi\rangle - \int_{\mathring{e}}[\![\langle f^*\omega\wedge\phi\rangle]\!].$$

However, keeping ϕ continuous, we could then apply differing transformations $h^{T_i} : T_i \to O(n)$ on either side of e, which we will collectively call h, and obtain another piecewise-smooth frame that we can evaluate the distributional curvature in. Since Ω is tensorial and ω obeys the gauge transformation law (2.3), we would get

$$\begin{aligned}
&\langle\langle (f\cdot h)^*\Omega_{\text{dist}}, \phi\rangle\rangle \\
&= \sum_{i=1,2}\int_{\mathring{T}_i}\langle \operatorname{Ad}({h^{T_i}}^{-1})({f^{T_i}}^*\Omega)\wedge\phi\rangle - \int_{\mathring{e}}\langle [\![\operatorname{Ad}(h^{-1})(f^*\omega) + h^{-1}dh]\!]\wedge\phi\rangle \\
&= \sum_{i=1,2}\int_{\mathring{T}_i}\langle {f^{T_i}}^*\Omega\wedge\operatorname{Ad}(h^{T_i})(\phi)\rangle - \int_{\mathring{e}}[\![\langle f^*\omega\wedge\operatorname{Ad}(h)(\phi)\rangle]\!] + \langle [\![h^{-1}dh]\!]\wedge\phi\rangle \\
&= \langle\langle f^*\Omega_{\text{dist}}, \operatorname{Ad}(h)(\phi)\rangle\rangle - \sum_{\mathring{e}\subset\mathring{M}}\int_{\mathring{e}}\langle [\![h^{-1}dh]\!]\wedge\phi\rangle.
\end{aligned}$$

If $f^*\Omega$ were a continuous $\mathfrak{so}(n)$-valued form, we would get
$\langle \operatorname{Ad}(h^{-1})(f^*\Omega)\wedge\phi\rangle = \langle f^*\Omega\wedge\operatorname{Ad}(h)(\phi)\rangle$. Guided by this, we define

$$\langle\langle \operatorname{Ad}(h^{-1})(f^*\Omega_{\text{dist}}), \phi\rangle\rangle := \langle\langle f^*\Omega_{\text{dist}}, \operatorname{Ad}(h)(\phi)\rangle\rangle. \tag{2.6}$$

As the Riemann curvature should be tensorial, it should always be true that changing the frame by h results in an adjoint action by h^{-1} on $f^*\Omega_{\text{dist}}$. For this to be true for continuous ϕ and discontinuous h, it must be the case that $[\![h^{-1}dh]\!] = 0$ along every codimension-1 face $e = T \cap T'$, so on e, $h^T = C_e h^{T'}$ for some constant matrix C_e. We will restrict ourselves to the case $C_e = I$, since otherwise it would not be possible to modify the transformation h so that it is the identity outside of a small neighborhood of x_0 but retains the jump condition, losing locality. What this means is that, if any particular frame f is asserted to be compatible, then it is reasonable to assert that in a small enough neighborhood U of a point $x \in \mathring{e}$, the set of compatible frames on U (meaning restrictions of compatible frames to U) must be contained in $\{(f|_U \cdot h) : h \text{ is piecewise-smooth and continuous}\}$. Therefore, piecewise-smooth frames are usually not compatible, and the set of piecewise-smooth compatible frames must all have the same jump conditions along codimension-1 polytopes—the matrix $\mu_e^T(\mu_e^{T'})^{-1}$ cannot depend on f. The choice $\mu_e^T(\mu_e^{T'})^{-1} = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix}$ is the simplest, and it is consistent with the case of continuous metrics and frames.

Enforcing the constraint $\mu_e^T = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \mu_e^{T'}$ on a frame precludes the possibility of it being continuous on the boundary of each polytope. For each codimension-2 polytope $p \subset M$, let E_p be a frame which is orthonormal with respect to $i_p^* g$, and for each pair of codimension-0 and codimension-1 polytopes T, e such that $p \subset e \subset T$, let $E_{p,e}^T$ be the extension of E_p to an orthonormal frame by appending the normal vector $\vec{\nu}$ which is orthogonal to p and points into e and the normal vector $\vec{n}$ which is orthogonal to e and makes $E_{p,e}^T$ right-hand oriented. If $p = e \cap e'$ and $e, e' \subset T$, then at each point $x \in p$,

$$E_{p,e}^T = E_{p,e'}^T \cdot \begin{bmatrix} I & 0 & 0 \\ 0 & \cos(\pm\theta_p^T) & -\sin(\pm\theta_p^T) \\ 0 & \sin(\pm\theta_p^T) & \cos(\pm\theta_p^T) \end{bmatrix} = E_{p,e'}^T \cdot \exp\left(\pm\theta_p^T w_n^{n-1}\right), \tag{2.7}$$

where θ_p^T is the dihedral angle between e and e' at x. Additionally, there exists an $O(n)$-

valued matrix $A^T_{p,e}$ such that

$$E^T_e = E^T_{p,e} \cdot A^T_{p,e}.$$

Note that since the final entries of E^T_e and $E^T_{p,e}$ are associated to normal vectors to e in the metric g^T, $A^T_{p,e} = \begin{bmatrix} A_{p,e} & 0 \\ 0 & \pm 1 \end{bmatrix}$, where $A_{p,e}$ does not depend on the polytope T. Because $E^{T'}_e$ has a normal vector that points in a different direction to that of E^T_e, but $E^{T'}_{p,e}$ does not, we also have $A^T_{p,e} = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} A^{T'}_{p,e} = A^{T'}_{p,e} \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix}$, and therefore

$$A^T_{p,e}\mu^T_e = A^{T'}_{p,e}\mu^{T'}_e.$$

Now suppose that for any Regge metric g, there exists a frame f that is continuous on each codimension-0 polytope T and satisfies the compatibility condition $\mu^T_e = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \mu^{T'}_e$. We will produce a contradiction. Let p be a codimension-2 polytope which is completely surrounded by codimension-0 polytopes $T_1, \ldots, T_k$, and let $e_i = T_i \cap T_{i+1}$ for $i = 1, 2, \ldots, k-1$ and $e_0 = e_k = T_k \cap T_1$. Without loss of generality, also assume that the ordering is chosen so that for each $i = 1, 2, 3, \ldots, k$, the signs in (2.7) are positive when $e = e_{i-1}$, $e' = e_i$, and $T = T_i$.

Then

$$f^{T_i} = E^{T_i}_{p,e_{i-1}} \cdot (A^{T_i}_{p,e_{i-1}}\mu^{T_i}_{e_{i-1}}) = E^{T_i}_{p,e_i} \cdot \left(\exp\left(\theta^{T_i}_p w^{n-1}_n\right) A^{T_i}_{p,e_{i-1}}\mu^{T_i}_{e_{i-1}}\right),$$

but also $f^{T_i} = E^{T_i}_{p,e_i} \cdot (A^{T_i}_{p,e_i}\mu^{T_i}_{e_i})$. Since the group action is free and $A^{T_i}_{p,e_{i-1}}\mu^{T_i}_{e_{i-1}} = A^{T_{i-1}}_{p,e_{i-1}}\mu^{T_{i-1}}_{e_{i-1}}$, this implies

$$\exp\left(\theta^{T_i}_p w^{n-1}_n\right) A^{T_{i-1}}_{p,e_{i-1}}\mu^{T_{i-1}}_{e_{i-1}} = A^{T_i}_{p,e_i}\mu^{T_i}_{e_i}. \tag{2.8}$$

Let Π^i_{i-1} be the linear transformation sending $f^{T_{i-1}}$ to f^{T_i}, meaning $\Pi^i_{i-1}(f^{T_{i-1}}_j) = f^{T_i}_j$ for each j. Clearly, $\Pi^1_k \circ \Pi^k_{k-1} \circ \cdots \circ \Pi^2_1 = I$. Let us express the transformation Π^i_{i-1} in the bases $E^{T_{i-1}}_{p,e_{i-1}}$ and $E^{T_i}_{p,e_i}$:

$$[\Pi^i_{i-1}]^{E^{T_i}_{p,e_i}}_{E^{T_{i-1}}_{p,e_{i-1}}} = A^{T_i}_{p,e_i}\mu^{T_i}_{e_i}(A^{T_{i-1}}_{p,e_{i-1}}\mu^{T_{i-1}}_{e_{i-1}})^{-1} = \exp\left(\theta^{T_i}_p w^{n-1}_n\right).$$

So we should get that

$$I = [\Pi^1_k \circ \cdots \circ \Pi^2_1]^{E^{T_1}_{p,e_1}}_{E^{T_1}_{p,e_1}} = \exp\left(\left(\sum_{i=1}^k \theta^{T_i}_p\right) w^{n-1}_n\right),$$

which implies $\sum_{i=1}^k \theta^{T_i}_p = 2m\pi$ for some integer m. However, in general, this sum can take any positive value if the metric is discontinuous at p. Therefore the frame f^T cannot always be continuous at p. This is the origin of the angle defect. In order to control the discontinuity of f^T as much as possible, we will restrict it to only rotate at a constant speed, and only in the plane orthogonal to p.

Summary of objects introduced. The preceding paragraphs introduce some objects that are used throughout the rest of the chapter. They are collected here for convenience.

Notation	Definition
E_p	An arbitrary orthonormal frame on the codimension-2 polytope p. Entries are called $\tau_1, \ldots, \tau_{n-2}$.
E_e	An arbitrary orthonormal frame on the codimension-1 polytope e. Entries are called $E_1, \ldots, E_{n-1}$.
$\vec{n}^T$	The outward-pointing g^T-normal vector to a face $e \subset T$. Usually called just $\vec{n}$ when T is implicitly known.
$\vec{\nu}$	The inward-pointing normal vector to a face $p \subset e$.
$E_{p,e}$	The g^e-orthonormal frame defined on p with entries $(\tau_1, \ldots, \tau_{n-2}, \vec{\nu})$
$A_{p,e}$	The map $p \to O(n-1)$ such that $E_e = E_{p,e} \cdot A_{p,e}$
E_e^T	The g^T-orthonormal frame defined on e with entries $(E_1, \ldots, E_{n-1}, \vec{n}^T)$.
$E_{p,e}^T$	The g^T-orthonormal frame defined on p with entries $(\tau_1, \ldots, \tau_{n-2}, \vec{\nu}, \pm\vec{n}^T)$, where the sign on $\vec{n}^T$ is chosen so that $E_{p,e}^T$ is positively oriented in M.
$A_{p,e}^T$	The map $p \to O(n)$ such that $E_e^T = E_{p,e}^T \cdot A_{p,e}^T$.
μ_e^T	The map $\mathring{e} \to O(n)$ such that $f^T\|_{\mathring{e}} = E_e^T \cdot \mu_e^T$.
θ_p^T	The map $p \to \mathbb{R}$ which measures the interior angle between the two faces $e, e' \subset T$ whose intersection is equal to p.

To ensure that we are still able to apply integration by parts, despite the fact that the frame is discontinuous at p, we will follow the strategy outlined in [5, p. 2] and require that f^T has some smoothness and continuity when pulled back to the *blow-up* B_T of T. The blow-up B_T of T is essentially a polytope that has one codimension-1 face for each codimension-d face of T with $d \geq 1$, with tangencies related to inclusion relations between the original faces. There is a corresponding *blow-down* map $\Phi^T : B_T \to T$ which restricts to a diffeomorphism $\Phi^T|_{\mathring{B}_T} : \mathring{B}_T \to \mathring{T}$. The blow-up has an exceptional set which we will call E_{B_T}, consisting of the faces of B_T of codimension ≥ 2, and it can be safely ignored for the purpose of Riemann integration. Some information on blow-ups of manifolds can be found in the appendix.

With all this in mind, we arrive at the definition of a compatible frame:

Definition 1. *A frame* $f = \bigsqcup_{T\subseteq M} f^T$, *where* $f^T : \mathring{T} \to \mathcal{F}_O(T)|_{\mathring{T}}$ *is a* C^2 *orthonormal frame for the metric* g^T, *is* compatible *if:*

1. *For each* $T \subseteq M$, *there exists a blow-up* B_T *(equipped with a blow-down map* $\Phi^T : B_T \to T$*) and a Lipschitz continuous map* $F^T : B_T \to \mathcal{F}_O(T)$ *such that* $F^T|_{B_T\backslash E_{B_T}} \in C^2(B_T\backslash E_{B_T};\mathcal{F}_O(T))$, $f^T \circ \Phi^T|_{\mathring{B}_T} = F^T|_{\mathring{B}_T}$, *and* $\pi \circ F^T = \Phi^T$. *We could also say that the following diagram is commutative (wherever the maps are defined):*

$$\begin{tikzcd}
B_T \arrow[r, "F^T"] \arrow[dr, "\Phi^T"] & \mathcal{F}_O(T) \arrow[d, shift left, "\pi"] \\
& T \arrow[u, shift left, "f^T"]
\end{tikzcd}$$

   This ensures that for each $e \subset T$, *there is a* C^2 *section* $f^T|_{\mathring{e}} : \mathring{e} \to \mathcal{F}_O(T)|_{\mathring{e}}$ *which continuously extends* f^T.

2. *For each* $e \subset T \subseteq M$, *let* $\mu_e^T : \mathring{e} \to O(n)$ *be the matrix such that* $E_e^T \cdot \mu_e^T = f^T|_{\mathring{e}}$, *and suppose* $e = T \cap T'$. *Then* $\mu_e^{T'} = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \mu_e^T$.

3. *For each* $p \subset T \subseteq M$, *let* $e \subset T$ *be the unique face meeting* p *such that the frame* $E_{p,e}^T$ *has an inward-pointing normal vector as its last entry. Then there exists an orientation-preserving embedding* $\psi_p^T : [0,1] \times \mathring{p} \to \overline{\Phi^{T^{-1}}(\mathring{p})}$, *where the orientation in* p *is induced by the orientation on* $e \subset T$, *and the orientation on* $\Phi^{T^{-1}}(\mathring{p})$ *is induced from* B_T, *such that* $\psi_p^T(\{0\} \times \mathring{p}) \subset \overline{\Phi^{T^{-1}}(\mathring{e})}$ *and* $\Phi^T \circ \psi_p^T(s,x) = x$. *Additionally, there exists a continuous function* $r_{p,e}^T : p \to \mathbb{R}$ *such that*

$$F^T(\psi_p^T(s,x)) = E_{p,e}^T(x) \cdot \left(\exp\left(s\ r_{p,e}^T(x) w_n^{n-1}\right) A_{p,e}^T(x) \mu_e^T(x)\right)$$

for all $x \in \mathring{p}$ *and all* $s \in [0, 1]$*. In other words, the multi-valuedness of* f^T *at codimension-2 faces is controlled so that* f^T *is continuous when expressed in cylindrical coordinates around* $x \in \mathring{p}$ *and rotates in the plane orthogonal to* p *at a rate depending only on* x*. There is no restriction on behavior near faces of higher codimension.*

4. *There exists a smooth metric* g_0 *and a smooth* g_0*-orthonormal frame* f_0 *such that there is a continuous homotopy of Regge metrics* $g(t)$ *with* $g(1) = g$ *and* $g(0) = g_0$*, and a homotopy of compatible frames* $f(t)$ *such that* $f(0) = f_0$*,* $f(1) = f$*, and* $f(t)$ *is* $g(t)$*-orthonormal and satisfies conditions 1, 2, and 3. The map* $(t, x) \mapsto F^T(t)(x)$ *also must vary continuously as a map* $[0, 1] \times B_T \to \mathcal{F}_{GL}(T)$*.*

As mentioned above, condition 1 makes it possible to integrate by parts even though the frame may have discontinuities on polytopes of codimension 2 or greater. The fact that F^T is Lipschitz and C^2 on the set $B_T \backslash E_{B_T}$ is necessary for the identity $dF^{T^*}\omega = F^{T^*}d\omega$ to hold and for $F^{T^*}\omega$ to be bounded and continuously extendable to $B_T \backslash E_{B_T}$. These are all necessary conditions for Stokes' Theorem to hold for the form $F^{T^*}(\omega \wedge \pi^*\phi)$ on B_T. Conditions 2 and 3 are necessary for the frame f to have some semblance of continuity and for the distributional curvature to transform like a tensor, as mentioned previously, and condition 4 states that compatible frames can be "smoothed out" without introducing singularities.

Remark 2. *We have ignored the question of whether the piecewise-smooth one-form* $f^*\omega$ *defined by* $f^*\omega^i_j|_{\mathring{T}} = \langle \nabla^T f^T_j, f^T_i \rangle_T$ *on* T *is "really" the connection form associated to the compatible frame* f*, despite the fact that* f *is discontinuous. Here is one way to answer that question: Since the pullback of* $f^*\theta^i$ *to every element interface* $e = T \cap T'$ *is single-valued, the distributional exterior derivative of* $f^*\theta^i$ *is simply its elementwise exterior derivative. Therefore the structure equations* $df^*\theta^i = -f^*\omega^i_j \wedge f^*\theta^j$ *hold in a distributional sense.*

2.2.2 The Integration by Parts Step

To perform integration by parts, we proceed in each codimension-0 polytope T by lifting the integrals in (2.5) to B_T and expanding out the inner products in coordinates. For now, we will suppress the ^{T} superscripts and focus on one term of the integral:

$$\begin{aligned}\int_{\mathring{T}} \langle f^*\omega \wedge d\phi \rangle &= \int_{\Phi(\mathring{B}_T)} f^*\omega^i_j \wedge d\phi^j_i \\ &= \int_{\mathring{B}_T} \Phi^* f^* \omega^i_j \wedge \Phi^* d\phi^j_i .\end{aligned}$$

Next we apply the facts that $f \circ \Phi = F$ on $\mathring{B}_T$, so $\Phi^* f^* = F^*$, and $\Phi = \pi \circ F$, so $\Phi^* = F^* \pi^*$. This yields

$$\int_{\mathring{B}_T} \Phi^* f^* \omega^i_j \wedge \Phi^* d\phi^i_j = \int_{\mathring{B}_T} F^*(\omega^i_j \wedge \pi^* d\phi^j_i).$$

The form $F^*(\omega^i_j \wedge \pi^* \phi^j_i)$ is C^1 and bounded on $\mathring{B}_T$, and summable on $\partial B_T \backslash E_{B_T}$, and $F^* d(\omega^i_j \wedge \pi^* \phi^j_i) = dF^*(\omega^i_j \wedge \pi^* \phi^j_i)$ is summable on $\mathring{B}_T$. Therefore the integration by parts theorem for Whitney manifolds [61, Theorem 18A] applies:

$$\int_{\mathring{B}_T} F^*(\omega^i_j \wedge \pi^* d\phi^j_i) = \int_{\mathring{B}_T} F^*(d\omega^i_j \wedge \pi^* \phi^j_i) - \int_{\partial B_T \backslash E_{B_T}} F^*(\omega^i_j \wedge \pi^* \phi^j_i).$$

The first term can be pulled back down to $\mathring{T}$, and the second term can be split up into integrals over the faces of B_T from different strata:

$$\begin{aligned}&\int_{\mathring{B}_T} F^*(d\omega^i_j \wedge \pi^* \phi^j_i) - \int_{\partial B_T \backslash E_{B_T}} F^*(\omega^i_j \wedge \pi^* \phi^j_i) \\ &= \int_{\mathring{T}} f^*(d\omega^i_j) \wedge \phi^j_i - \sum_{d=1}^{n} \int_{\Phi^{-1}(S_d(T))} F^*(\omega^i_j \wedge \pi^* \phi^j_i).\end{aligned}$$

For $d > 2$, the integrals over $\Phi^{-1}(\mathring{\Delta}_d)$ vanish for any codimension-d face $\mathring{\Delta}_d \subset S_d(T)$. Indeed, ϕ^j_i is an $(n-2)$-form, so its trace $i^*_{\mathring{\Delta}_d} \phi^j_i$ vanishes on any $\mathring{\Delta}_d$ with $d > 2$. Since $i^*_{\Phi^{-1}(\mathring{\Delta}_d)} F^* \pi^* = i^*_{\Phi^{-1}(\mathring{\Delta}_d)} \Phi^* = \Phi^* i^*_{\mathring{\Delta}_d}$, we have $i^*_{\Phi^{-1}(\mathring{\Delta}_d)} F^* \pi^* \phi^i_j = \Phi^* i^*_{\mathring{\Delta}_d} \phi^j_i = 0$. Therefore the integral of $F^*(\omega^i_j \wedge \pi^* \phi^j_i)$ over $\Phi^{-1}(S_d(T)) = \Phi^{-1}(\bigcup \mathring{\Delta}_d)$ is equal to zero.

Additionally, since there is a smooth (in each component) section $f|_{S_1(T)}$ of $\mathcal{F}_O(T)|_{S_1(T)}$ that continuously extends f (by condition 1),

$$\int_{\Phi^{-1}(S_1(T))} F^*(\omega^i_j \wedge \pi^*\phi^j_i) = \int_{S_1(T)} f|^*_{S_1(T)}(\omega^i_j \wedge \pi^*\phi^j_i) = \int_{S_1(T)} f|^*_{S_1(T)}\omega^i_j \wedge \phi^j_i.$$

Here, we have implicitly used the fact that $\Phi|_{\Phi^{-1}(S_1(T))} : \Phi^{-1}(S_1(T)) \to S_1(T)$ is a diffeomorphism on each component; see Appendix 5.1.

Plugging this back into (2.5), we get

$$\begin{align*}
\langle\langle f^*\Omega_{\text{dist}}, \phi\rangle\rangle &= \sum_{T\subseteq M} \int_{\mathring{T}} f^{T^*}(d\omega^i_j + \omega^i_k \wedge \omega^k_j) \wedge \phi^j_i - \int_{S_1(T)} f^T|^*_{S_1(T)}\omega^i_j \wedge \phi^j_i \tag{2.9}\\
&\quad - \int_{\Phi^{T^{-1}}(S_2(T))} F^{T^*}(\omega^i_j \wedge \pi^*\phi^j_i)\\
&= \sum_{T\subseteq M} \int_{\mathring{T}} \langle f^{T^*}(d\omega + \frac{1}{2}[\omega,\omega]) \wedge \phi\rangle - \sum_{\mathring{e}\subset\mathring{M}} \int_{\mathring{e}} \langle \llbracket f^T|^*_{\mathring{e}}\omega \rrbracket \wedge \phi\rangle \tag{2.10}\\
&\quad - \sum_{\mathring{e}\subset\mathring{M}} \sum_{T\supset p} \int_{\Phi^{T^{-1}}(\mathring{p})} F^{T^*}\langle \omega \wedge \pi^*\phi\rangle.
\end{align*}$$

For the integrals over codimension-1 polytopes, let $\mathrm{I\!I}^T_e \in \Omega^1(e, \mathfrak{so}(n))$ be the form defined by $(\mathrm{I\!I}^T_e)^i_j = 0$ for all $1 \le i, j \le n-1$ and $(\mathrm{I\!I}^T_e)^n_i = \langle \nabla(E^T_e)_i, \vec{n}\rangle$ for $1 \le i \le n$. We will use it to represent the second fundamental form of e. Also let $\tilde{\omega}^T_e = (E^T_e)^*\omega$, that is, the full connection form for the frame E^T_e. Then the gauge transformation law (2.3) tells us that $f^T|^*_{\mathring{e}}\omega = \mathrm{Ad}((\mu^T_e)^{-1})(\tilde{\omega}^T_e) + (\mu^T_e)^{-1}d\mu^T_e$.

Since the first $n-1$ vectors in E^T_e do not depend on T, there exists an $\mathfrak{so}(n-1)$-valued form J_e that does not depend on T so that $\tilde{\omega}^T_e - \mathrm{I\!I}^T_e = \begin{bmatrix} J_e & 0 \\ 0 & 0 \end{bmatrix}$. For interior edges $e = T \cap T'$, $(\mu^T_e)^{-1}d\mu^T_e = (\mu^{T'}_e)^{-1}d\mu^{T'}_e$, and the following equation gives us the jump in ω:

$$\begin{aligned}
\llbracket f^T|^*_{\mathring{e}}\omega \rrbracket &= \mathrm{Ad}((\mu^T_e)^{-1})(\tilde{\omega}^T_e) - \mathrm{Ad}((\mu^{T'}_e)^{-1})(\tilde{\omega}^{T'}_e)\\
&= \mathrm{Ad}((\mu^T_e)^{-1})\left(\tilde{\omega}^T_e - \mathrm{Ad}\left(\begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix}\right)(\tilde{\omega}^{T'}_e)\right)\\
&= \mathrm{Ad}((\mu^T_e)^{-1})(\mathrm{I\!I}^T_e + \mathrm{I\!I}^{T'}_e).
\end{aligned}$$

This could still be interpreted as a jump in second fundamental form, since the two fundamental forms are being evaluated with normal vectors that point in opposite directions. We will therefore denote $\llbracket \mathrm{II}_e \rrbracket := \mathrm{II}_e^T + \mathrm{II}_e^{T'}$ when $e = T \cap T'$. The integrals over codimension-1 polytopes then read

$$\int_{\mathring{e}} \langle \llbracket f^T|_{\mathring{e}}^* \omega \rrbracket \wedge \phi \rangle = \int_{\mathring{e}} \langle \mathrm{Ad}((\mu_e^T)^{-1})(\llbracket \mathrm{II}_e \rrbracket) \wedge \phi \rangle. \tag{2.11}$$

While this expression involves a codimension-0 polytope T of which e is a face, the expression is not actually dependent on T, because both the sign of the integrand and the orientation of the integral change when T and T' are interchanged. The only caveat is that the integral must be evaluated using the induced orientation of e from T.

Note also that we have made key use of the fact that $i_{\mathring{e}}^* \phi = 0$ for $e \subset \partial M$, so there is no issue with the boundary terms here. See Section 2.4 for some discussion on how things change if ϕ is allowed to not pull back to zero on ∂M.

2.2.3 Derivation of the Angle Defect

Expanding out the last term on the right of 2.10 takes more work. First, we will use condition 3 on f to change the integrals over ${\Phi^T}^{-1}(\mathring{p})$ to integrals over $(0,1) \times \mathring{p}$:

$$\int_{{\Phi^T}^{-1}(\mathring{p})} F^{T^*}(\omega_j^i \wedge \pi^* \phi_i^j) = \int_{(0,1)\times\mathring{p}} \psi_p^{T^*} F^{T^*}(\omega_j^i \wedge \pi^* \phi_i^j).$$

The form $\psi_p^{T^*} F^{T^*} \pi^* \phi_i^j = \psi_p^{T^*} \Phi^{T^*} \phi_i^j$ is basic for the fiber bundle $q : [0,1] \times p \to p$, i.e. $\psi_p^{T^*} \Phi^{T^*} \phi_i^j = q^* i_p^* \phi_i^j$. Then Fubini's theorem can be applied to this integral:

$$\int_{(0,1)\times\mathring{p}} \psi_p^{T^*} F^{T^*}(\omega_j^i \wedge \pi^* \phi_i^j) = \int_{\mathring{p}} \left(\int_{(0,1)\times\{x\}} \psi_p^{T^*} F^{T^*} \omega_j^i \right) \phi_i^j.$$

Let $\xi_p^T(x) := \eta(dF^T \circ d\psi_p^T(\frac{\partial}{\partial s}|_{(x,s)}))$ be the $\mathfrak{so}(n)$-valued function representing counterclockwise rotation around p at the angular speed $r_{p,e}^T(x)$. The fact that ξ is independent of s is a consequence of condition 3. Then since $\omega(X) = \eta(X)$ for any vertical vector X, the integral can be made simpler:

$$\begin{aligned}\int_{\mathring{p}}\left(\int_{(0,1)\times\{x\}}\psi_p^{T^*}F^{T^*}\omega_j^i\right)\phi_i^j &= \int_{\mathring{p}}\left(\int_0^1 \omega_j^i\left(dF^T\circ d\psi_p^T\left(\frac{\partial}{\partial s}\right)\right)ds\right)\phi_i^j\\ &= \int_{\mathring{p}}\left(\int_0^1 (\xi_p^T)_j^i(x)ds\right)\phi_i^j = \int_{\mathring{p}}(\xi_p^T)_j^i\phi_i^j.\end{aligned}$$

This means we can simplify the final term of (2.10) into a single integral over $\mathring{p}$:

$$\sum_{T\supset p}\int_{\Phi^{T^{-1}}(\mathring{p})}F^{T^*}(\omega_j^i\wedge\pi^*\phi_i^j) = \int_{\mathring{p}}\langle\sum_{T\supset p}\xi_p^T\wedge\phi\rangle. \tag{2.12}$$

The vector $\xi_p^T\in\mathfrak{so}(n)$ can be explicitly calculated, using condition 3 of the compatible frame, as $\mathrm{Ad}((A_{p,e}^T\mu_e^T)^{-1})(r_{p,e}^T w_n^{n-1})$. The sum of these terms is the derivative of

$$G_p(s) := \prod_{j=k}^{1}\mathbf{Ad}((A_{p,e_j}^{T_j}\mu_{e_j}^{T_j})^{-1})\left(\exp\left(s\ r_{p,e_j}^{T_j}w_n^{n-1}\right)\right)$$

at $s=0$, where T_j and e_j are some enumeration of the codimension-0 and codimension-1 faces surrounding p such that for each $j=1,\dots,k-1$, $T_j\cap T_{j+1}=e_{j+1}$, the rotation from $E_{p,e_j}^{T_j}$ to $E_{p,e_{j+1}}^{T_j}$ is in the counterclockwise direction for each $j=1,\dots,k-1$, and k is the number of codimension-1 faces incident to p. Another way to characterize this enumeration is that $E_{p,e_j}^{T_j}$ has an inward-pointing normal vector as its last element and $E_{p,e_{j+1}}^{T_j}$ has an outward-pointing normal vector as its last element. If p is an interior polytope, then we also require that the preceding statements hold for $j=k$, where $T_{k+1}:=T_1$ and $e_{k+1}:=e_1$.

Note that above, the multiplication is arranged so that $j=k$ is the leftmost factor and $j=1$ is the rightmost factor. The order of multiplication here technically doesn't matter, as the derivative of $G_p(s)$ at $s=0$ is the same regardless of the ordering of multiplications, but this is the convention we chose.

Next we will use algebraic manipulations much like those in equation (2.8) to derive an expression for $A_{p,e_{j+1}}^{T_{j+1}}\mu_{e_{j+1}}^{T_{j+1}}(A_{p,e_j}^{T_j}\mu_{e_j}^{T_j})^{-1}$.

For $x\in p$, condition 3 on the compatible frame (and the fact that F^{T_j} is continuous) implies

$$f^{T_j}|_{e_{j+1}}(x) = F^{T_j}(\psi_p^{T_j}(1,x)) = E_{p,e_j}^{T_j}\cdot\exp\left(r_{p,e_j}^{T_j}(x)w_n^{n-1}\right)A_{p,e_j}^{T_j}\mu_{e_j}^{T_j}.$$

However, we also know that $f^{T_j}|_{e_{j+1}} = E^{T_j}_{p,e_{j+1}} \cdot A^{T_j}_{p,e_{j+1}} \mu^{T_j}_{e_{j+1}}$, so because the rotation from $E^{T_j}_{p,e_j}$ to $E^{T_j}_{p,e_{j+1}}$ is in the counterclockwise direction, we have $E^{T_j}_{p,e_{j+1}} = E^{T_j}_{p,e_j} \cdot \exp\left(\theta^{T_j}_p w^{n-1}_n\right)$, so we can use the fact that the group action is free on each fiber of $\mathcal{F}_O(T_j)$ to derive

$$\exp\left(r^{T_j}_{p,e_j} w^{n-1}_n\right) A^{T_j}_{p,e_j} \mu^{T_j}_{e_j} = \exp\left(\theta^{T_j}_p w^{n-1}_n\right) A^{T_j}_{p,e_{j+1}} \mu^{T_j}_{e_{j+1}}.$$

Lastly we can apply the fact that $A^{T_j}_{p,e_{j+1}} \mu^{T_j}_{e_{j+1}} = A^{T_{j+1}}_{p,e_{j+1}} \mu^{T_{j+1}}_{e_{j+1}}$ to derive

$$A^{T_{j+1}}_{p,e_{j+1}} \mu^{T_{j+1}}_{e_{j+1}} (A^{T_j}_{p,e_j} \mu^{T_j}_{e_j})^{-1} = \exp\left((r^{T_j}_{p,e_j} - \theta^{T_j}_p) w^{n-1}_n\right). \tag{2.13}$$

Then $G_p(s) = (A^{T_k}_{p,e_k} \mu^{T_k}_{e_k})^{-1} \exp\left(\gamma(s) w^{n-1}_n\right) A^{T_1}_{p,e_1} \mu^{T_1}_{e_1}$, where

$$\gamma(s) := s r^{T_k}_{p,e_k} + \sum_{j=1}^{k-1} (1+s) r^{T_j}_{p,e_j} - \theta^{T_j}_p.$$

For interior polytopes, this can be simplified with one more application of (2.13) to obtain the equivalent expression

$$G_p(s) = \mathbf{Ad}((A^{T_k}_{p,e_k} \mu^{T_k}_{e_k})^{-1}) \left(\exp\left[(\gamma(s) + r^{T_k}_{p,e_k} - \theta^{T_k}_p) w^{n-1}_n\right]\right). \tag{2.14}$$

Therefore for interior polytopes, $G_p(0) = I$ implies that $\gamma(0) + r^{T_k}_{p,e_k} - \theta^{T_k}_p = 2\pi m$ for an integer m. The integer m must be continuous with respect to continuous deformations of g and f. Thus, using the final condition for compatible frames and the fact that $r^T_{p,e} = 0$ for continuous frames and continuous metrics, we get $m = -1$. Therefore $\sum_{j=1}^k r^{T_j}_{p,e_j} = -\Theta_p$, where $\Theta_p := 2\pi - \sum_{j=1}^k \theta^{T_j}_p$ is the angle defect of p at x. Lastly we note that, due to equation (2.14),

$$\left.\frac{\partial}{\partial s}\right|_{s=0} G_p(s) = \mathrm{Ad}\left((A^{T_k}_{p,e_k} \mu^{T_k}_{e_k})^{-1}\right) \left(\sum_{j=1}^k r^{T_j}_{p,e_j} w^{n-1}_n\right),$$

so we finally get

$$\int_{\mathring{p}} \langle \sum_{T \supset p} \xi^T_p \wedge \phi \rangle = -\int_{\mathring{p}} \Theta_p \langle \mathrm{Ad}((A^{T_k}_{p,e_k} \mu^{T_k}_{e_k})^{-1})(w^{n-1}_n) \wedge \phi \rangle. \tag{2.15}$$

The left-hand side of this equation clearly doesn't depend on the particular enumeration T_j we picked, so we will omit the subscript $_k$ from now on and remember that although the expression on the right-hand side involves particular codimension-0 and codimension-1 polytopes T and e of which p is a face, the expression is actually independent of the particular choice. The only caveat is that the integral needs to be evaluated with the orientation on p induced from e, and e needs to be chosen so that the last entry of $E^T_{p,e}$ is an inward-pointing normal vector; otherwise the sign on Θ_p would need to flip.

2.2.4 Properties of the Distributional Curvature

Combining (2.10), (2.11), (2.12), and (2.15), we arrive at our final expression for the distributional curvature:

Theorem 1. *Suppose* $f = \bigsqcup_{T\subseteq M} f^T$ *satisfies all of the compatibility conditions in Definition 1 and* $\phi \in C^\infty_c\Omega^{n-2}(M;\mathfrak{so}(n))$ *is a smooth compactly supported* $\mathfrak{so}(n)$*-valued* $(n-2)$*-form which vanishes when pulled back to* ∂M*. Then*

$$\langle\langle f^*\Omega_{\text{dist}}, \phi\rangle\rangle = \sum_{T\subseteq M}\int_{\mathring{T}}\langle {f^T}^*\Omega\wedge\phi\rangle - \sum_{\mathring{e}\subset\mathring{M}}\int_{\mathring{e}}\langle \mathrm{Ad}((\mu^T_e)^{-1})([\![\mathrm{I\!I}_e]\!])\wedge i^*_{\mathring{e}}\phi\rangle$$
$$+\sum_{\mathring{p}\subset\mathring{M}}\int_{\mathring{p}}\Theta_p\langle \mathrm{Ad}((A^T_{p,e}\mu^T_e)^{-1})(w^{n-1}_n)\wedge i^*_{\mathring{p}}\phi\rangle. \qquad (2.16)$$

Furthermore, the distributional curvature is tensorial, in the following sense: if h *is a continuous piecewise-smooth map* $M\to O(n)$*, then* $f\cdot h$ *is a compatible frame and*

$$\langle\langle (f\cdot h)^*\Omega_{\text{dist}}, \phi\rangle\rangle = \langle\langle \mathrm{Ad}(h^{-1})(f^*\Omega_{\text{dist}}), \phi\rangle\rangle.$$

Proof. The proof of the expression for the distributional curvature was already carried out in subsections 2.1-2.3, and it is clear that $f\cdot h$ satisfies all the compatibility conditions if h is continuous and piecewise-smooth, so all that remains to be proven is the tensoriality. First, we will expand out the expression for the distributional curvature in the frame $\hat{f} = f\cdot h$. Note that since $f = E^T_e\cdot\mu^T_e$, $\hat{\mu}^T_e = \mu^T_e h$, so the expression is not hard to evaluate:

$$
\begin{aligned}
&\langle\langle \hat{f}^*\Omega_{\mathrm{dist}}, \phi\rangle\rangle\\
&= \sum_{T\subseteq M}\int_{\mathring{T}}\langle (f^T\cdot h)^*\Omega\wedge\phi\rangle - \sum_{\mathring{e}\subset\mathring{M}}\int_{\mathring{e}}\langle \mathrm{Ad}((\mu_e^T h)^{-1})([\![\mathrm{I\!I}_e]\!])\wedge i^*_{\mathring{e}}\phi\rangle\\
&\quad + \sum_{\mathring{p}\subset\mathring{M}}\int_{\mathring{p}}\Theta_p\langle \mathrm{Ad}((A^T_{p,e}\mu^T_e h)^{-1})(w_n^{n-1})\wedge i^*_{\mathring{p}}\phi\rangle\\
&= \sum_{T\subseteq M}\int_{\mathring{T}}\langle \mathrm{Ad}(h^{-1})({f^T}^*\Omega)\wedge\phi\rangle - \sum_{\mathring{e}\subset\mathring{M}}\int_{\mathring{e}}\langle \mathrm{Ad}(h^{-1})(\mathrm{Ad}((\mu_e^T)^{-1})([\![\mathrm{I\!I}_e]\!]))\wedge i^*_{\mathring{e}}\phi\rangle\\
&\quad + \sum_{\mathring{p}\subset\mathring{M}}\int_{\mathring{p}}\Theta_p\langle \mathrm{Ad}(h^{-1})(\mathrm{Ad}((A^T_{p,e}\mu^T_e)^{-1})(w_n^{n-1}))\wedge i^*_{\mathring{p}}\phi\rangle\\
&= \sum_{T\subseteq M}\int_{\mathring{T}}\langle {f^T}^*\Omega\wedge\mathrm{Ad}(h)(\phi))\rangle - \sum_{\mathring{e}\subset\mathring{M}}\int_{\mathring{e}}\langle \mathrm{Ad}((\mu_e^T)^{-1})([\![\mathrm{I\!I}_e]\!])\wedge \mathrm{Ad}(h)(i^*_{\mathring{e}}\phi))\rangle\\
&\quad + \sum_{\mathring{p}\subset\mathring{M}}\int_{\mathring{p}}\Theta_p\langle \mathrm{Ad}((A^T_{p,e}\mu^T_e)^{-1})(w_n^{n-1})\wedge \mathrm{Ad}(h)(i^*_{\mathring{p}}\phi))\rangle\\
&= \langle\langle f^*\Omega_{\mathrm{dist}}, \mathrm{Ad}(h)(\phi)\rangle\rangle\\
&= \langle\langle \mathrm{Ad}(h^{-1})(f^*\Omega_{\mathrm{dist}}), \phi\rangle\rangle.
\end{aligned}
$$

□

One nice consequence of Equation (2.16) is that, if $K\subseteq M$ is compact, then there exists a number $C_K<\infty$ such that for all test forms ϕ having support contained in K, $|\langle\langle f^*\Omega_{\mathrm{dist}},\phi\rangle\rangle| < C_K\sup_{x\in M}\|\phi(x)\|_g$. In other words, $f^*\Omega_{\mathrm{dist}}$ is a distribution, or more specifically a current, of order 0. Additionally, since the curvature depends only on integrals of ϕ over submanifolds of codimension ≤ 2, a coordinate-dependent H^2 norm can be chosen on K, and by the Rellich theorem, $\|f^*\Omega_{\mathrm{dist}}\|_{H^{-2}(K)}<\infty$.

Another observation we can make is that the right-hand side of (2.16) evaluates to zero if ϕ is instead $\mathrm{sym}_{n\times n}(\mathbb{R})$-valued, since all of the integrals are against $\mathfrak{so}(n)$-valued forms. This supports the intuitive notion that $f^*\Omega_{\mathrm{dist}}$ is "$\mathfrak{so}(n)$-valued", as $\mathfrak{so}(n)$ and $\mathrm{sym}_{n\times n}(\mathbb{R})$ are orthogonal under the bilinear product we are using.

The space of test forms can also be expanded to include discontinuous forms of a particular type.

Definition 2. *Consider a compactly supported* $\mathfrak{so}(n)$*-valued* $(n-2)$*-form* ϕ *which is* C^2 *on* $\mathring{T}$ *for each codimension-0 polytope* $T \subseteq M$*. Such a form will be called a* compatible $\mathfrak{so}(n)$-valued test form *if the following conditions are satisfied:*

1. *For each* $T \subseteq M$ *there exists a unique Lipschitz continuous extension of* ${\Phi^T}^*\phi|_{\mathring{T}}$ *to* B_T*, which will be called* $\tilde{\phi}^T \in C^0\Omega^{n-2}(B_T;\mathfrak{so}(n))$*. In other words,* ${\Phi^T}^*\phi|_{\mathring{T}}$ *has uniformly bounded first derivatives in any coordinate chart. This implies* $i^*_{\mathring{e}}\phi|_T := {\Phi^T}^{-*} i^*_{{\Phi^T}^{-1}(\mathring{e})}\tilde{\phi}^T$ *is a well-defined continuous extension of* $\phi|_{\mathring{T}}$*. It also implies that if* $d\Phi^T(v) = 0$*, then* $v \lrcorner \tilde{\phi}^T = 0$ *(this is relevant only on the boundary of* B_T*).*

2. *If* $e = T \cap T'$ *is a codimension-1 interface, then* $i^*_{\mathring{e}}\phi|_T = i^*_{\mathring{e}}\phi|_{T'}$*. In other words,* $[\![i^*_{\mathring{e}}\phi]\!] = 0$*.*

3. *The form* ${\psi_p^T}^*\tilde{\phi}^T$ *is basic for the fibration* $q : [0,1] \times p \to p$*. More precisely, if* $\chi^T_{s,p}(x) := \psi^T_p(s,x)$*, Then* ${\chi^T_{s,p}}^*\tilde{\phi}^T = {\chi^T_{0,p}}^*\tilde{\phi}^T$ *for any* $s \in [0,1]$*. In other words,* $i^*_{\mathring{p}}\phi|_T$ *is well-defined and continuous on* p*, although* ϕ *itself may be discontinuous at* p*.*

4. *If* $\Delta_d \subset \partial M$*, then* $i^*_{{\Phi^T}^{-1}(\mathring{\Delta}_d)}\tilde{\phi}^T = 0$ *for each* $T \supset \Delta_d$*. In other words,* $i^*_{\partial M}\phi = 0$*.*

The set of compatible test forms for the manifold M *with mesh* $\mathcal{T}$ *will be called* $\mathcal{C}(\mathcal{T}, M)$*.*

If ϕ is a compatible $\mathfrak{so}(n)$-valued test form, then the proofs in Sections 2.2.2-2.2.3 remain valid with the small modification of using $\tilde{\phi}^T$ in place of ${\Phi^T}^*\phi$. Note that items (2) and (3) together imply that $i^*_{\mathring{p}}\phi = {\chi^T_{s,p}}^*\tilde{\phi}^T$ is well-defined independently of s and T. A notable example of compatible test forms are the forms $\phi_{ij} \otimes \star f^*(\theta^i \wedge \theta^j)$, where each $\phi_{ij} : M \to \mathfrak{so}(n)$, $i, j = 1, 2, \ldots, n$, is a compactly supported continuous map that is C^2 on the interior of each codimension-0 polytope and vanishes on ∂M. The map $\star : \Omega^2(M) \to \Omega^{n-2}(M)$ is, on each codimension-0 polytope T, the Hodge star operator associated to the metric g^T.

2.2.5 Removing Frame-Dependency

While useful for analysis, (2.16) is inconvenient for many applications because it requires knowledge of a specific compatible frame, which are not easy to construct or represent. The purpose of this subsection is to remove this dependency by expressing the distributional curvature in terms of endomorphism-valued forms. The cost is that the test forms must be discontinuous and metric-dependent.

First, we can define a symmetric nondegenerate bilinear form on the vector space $\mathrm{End}(T_xM) = T_xM \otimes T_x^*M$ by setting

$$\langle \alpha \otimes v, \beta \otimes w \rangle := \alpha(w)\beta(v) \tag{2.17}$$

for $\alpha, \beta \in T_x^*(M)$, $v, w \in T_xM$ and extending multilinearly. Expressed in a basis $\{F_i\}_{i=1}^n$ and its dual basis $\{F^i\}_{i=1}^n$, this would be $\langle A^i_j F_i \otimes F^j, B^k_l F_k \otimes F^l \rangle = A^i_j B^j_i$. Therefore, picking any basis for T_xM gives a linear isometry between $\mathfrak{gl}(n)$ and $\mathrm{End}(T_xM)$, by setting $F^j(\hat{\phi}^T(F_i)) := \phi^j_i$. The left-hand side of this equation could be more succinctly written $([\hat{\phi}]_F)^j_i$, where $[\hat{\phi}]_F$ is the matrix representation of the linear map $\hat{\phi} : T_xM \to T_xM$ in the basis F. In particular, for $y \in B_T$, the orthonormal basis $F^T(y)$ gives a special isometry $\Psi^T_y : \mathfrak{gl}(n) \to \mathrm{End}(T_{\Phi^T(y)}M)$.

This pointwise isometry in turn defines a map Ψ_f from piecewise-continuous $\mathfrak{so}(n)$-valued $(n-2)$-forms to piecewise-continuous $\mathrm{End}(TM)$-valued $(n-2)$-forms, by setting $\Psi_f(\phi)|_x$ equal to $\Psi^T_{\Phi^{T^{-1}}(x)}(\phi|_x)$ if x is in $\mathring{T}$. Clearly, because Φ^T_y is invertible for all $y \in \mathring{B}_T$ and Φ^T is a diffeomorphism when restricted to $\mathring{B}_T$, Ψ_f is a bijection, with inverse given by $\Psi_f^{-1}(\hat{\phi})|_x := (\Psi^T_{\Phi^{T^{-1}}(x)})^{-1}(\hat{\phi}|_x)$.

Definition 3. *Let f be a compatible frame. A* compatible $\mathrm{End}(TM)$-valued test form *is the image of a compatible $\mathfrak{so}(n)$-valued test form (see Definition 2) under the map Ψ_f. The space of compatible* $\mathrm{End}(TM)$*-valued test forms is denoted $\mathcal{A}(f, \mathcal{T}, M) := \Psi_f(\mathcal{C}(\mathcal{T}, M))$.*

Several properties of $\mathrm{End}(TM)$-valued test forms make them more suited to frame-independent computations.

Let $y \in B_T$ and $h \in O(n)$, and let E be the basis of $T_{\Phi^T(y)}M$ determined by $F = E \cdot h$. Then the adjoint action of h changes $\Psi_y^T(u)$ by essentially changing the basis it is represented in:

$$\begin{aligned} F^j\left(\Psi_y^T(\mathrm{Ad}(h)(u))(F_i)\right) &= h_l^j u_k^l (h^{-1})_i^k \\ &= h_l^j F^l\left(\Psi_y^T(u)(F_k)\right)(h^{-1})_i^k \\ &= h_l^j F^l\left(\Psi_y^T(u)(F_k(h^{-1})_i^k)\right) \\ &= E^j\left(\Psi_y^T(u)(E_i)\right). \end{aligned} \tag{2.18}$$

Since F^T is continuous and Lipschitz on B_T and ${\Phi^T}^*\phi|_{\mathring{T}}$ has a unique Lipschitz extension $\tilde{\phi}^T$ on B_T, there exists a unique Lipschitz extension of ${\Phi^T}^*\hat{\phi}|_{\mathring{T}}$ to B_T, which at each point $y \in B_T$ is equal to $\Psi_y^T(\tilde{\phi}^T|_y)$. We will use $\hat{\phi}^T$ to refer to this extension of ${\Phi^T}^*\hat{\phi}|_{\mathring{T}}$. Similarly to the case for compatible $\mathfrak{so}(n)$-valued test forms, the pullback $i_{\mathring{e}}^*\hat{\phi}|_T := {\Phi^T}^{-*} i^*_{{\Phi^T}^{-1}(\mathring{e})}\hat{\phi}^T$ is a well defined continuous extension of $\hat{\phi}|_{\mathring{T}}$.

The two key properties we will need about compatible $\mathrm{End}(TM)$-valued test forms are proved in the following lemma.

Lemma 1. *Let $\hat{\phi} = \Psi_f(\phi)$ be a compatible $\mathrm{End}(TM)$-valued test form. Then the following change of basis equation holds for each $T, T' \subseteq M$ and $e = T \cap T'$:*

$$[i_{\mathring{e}}^*\hat{\phi}|_T]_{E_e^T} = \mathrm{Ad}\left(\begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix}\right)\left([i_{\mathring{e}}^*\hat{\phi}|_{T'}]_{E_e^{T'}}\right). \tag{2.19}$$

Additionally, the following equation holds for each $p \subset e \subset T$ such that $E_{p,e}^T$ has an inward-pointing normal vector as its lastly entry:

$$[{\chi_{s,p}^T}^*\hat{\phi}^T]_{E_{p,e}^T} = \mathrm{Ad}\left(\exp\left(s\ r_{p,e}^T w_n^{n-1}\right)\right)\left([{\chi_{0,p}^T}^*\hat{\phi}^T]_{E_{p,e}^T}\right). \tag{2.20}$$

Here, $\chi_{s,p}^T$ is the map defined within condition 3 of Definition 2.

Proof. By (2.18), at points $x \in e = T \cap T'$, we have on T that $\mathrm{Ad}((\mu_e^T)^{-1})([i_{\mathring{e}}^*\hat{\phi}|_T]_{E_e^T}) = i_{\mathring{e}}^*\phi = \mathrm{Ad}((\mu_e^{T'})^{-1})([i_{\mathring{e}}^*\hat{\phi}|_{T'}]_{E_e^{T'}})$, since ϕ is compatible. However, because $\mu_e^{T'} = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \mu_e^T$, we have

$$\mathrm{Ad}((\mu_e^T)^{-1}) \left([i_{\mathring{e}}^*\hat{\phi}|_T]_{E_e^T} - \mathrm{Ad}\left(\begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \right) \left([i_{\mathring{e}}^*\hat{\phi}|_{T'}]_{E_e^{T'}} \right) \right) = 0$$

By applying $\mathrm{Ad}(\mu_e^T)$ to both sides of this equation, we get Equation (2.19).

Similarly, the form $[i_{\mathring{p}}^*\hat{\phi}^T]_{F^T} = i_{\mathring{p}}^*\phi$ is well-defined, in the sense that ${\chi_{s,p}^T}^*\tilde{\phi}^T$ does not depend on s or T. Below we will use the shorthand $F(s,x) = F^T \circ \psi_p^T(s,x)$.

By (2.18) and condition 3 of compatible frames, for any $s \in [0,1]$, we have

$$[{\chi_{s,p}^T}^*\hat{\phi}^T]_{E_{p,e}^T} = \mathrm{Ad}\left(\exp\left(s\ r_{p,e}^T w_n^{n-1}\right) A_{p,e}^T \mu_e^T\right) \left([{\chi_{s,p}^T}^*\hat{\phi}^T]_{F(s,\cdot)}\right)$$

and

$$[{\chi_{0,p}^T}^*\hat{\phi}^T]_{E_{p,e}^T} = \mathrm{Ad}(A_{p,e}^T \mu_e^T)([{\chi_{0,p}^T}^*\hat{\phi}^T]_{F(0,\cdot)}).$$

Since ${\chi_{s,p}^T}^*\tilde{\phi}^T = {\chi_{0,p}^T}^*\tilde{\phi}^T$ is one of the defining characteristics of a compatible $\mathfrak{so}(n)$-valued test form and $[{\chi_{s,p}^T}^*\hat{\phi}^T]_{F(s,\cdot)} = {\chi_{s,p}^T}^*\tilde{\phi}^T$, the second equation can be substituted into the first to obtain Equation (2.20).

□

A more intuitive way to understand what this lemma gives us is: rather than $\hat{\phi}$ itself being continuous, we get that the matrix representations of $\hat{\phi}$ in special frames adapted to the mesh are only allowed to have discontinuities in a special way, and except for the dependence on $r_{p,e}^T$, the discontinuities do not depend on the frame f.

Just as for compatible test forms, an important class of compatible $\mathrm{End}(TM)$-valued test forms is given by the forms $\phi_{lij}^k(f_k \otimes f^*\theta^l) \otimes \star f^*(\theta^i \wedge \theta^j)$, where the coefficients ϕ_{lij}^k are continuous compactly supported functions on M which are C^2 on the interior of each codimension-0 polytope and vanish on ∂M and which alternate in the k,l indices. These are exactly the images of the forms $\phi'_{ij} \otimes \star f^*(\theta^i \wedge \theta^j)$ under the map Ψ_f, where $\phi'_{ij} : M \to \mathfrak{so}(n)$

has the same continuity and smoothness as the maps ϕ^k_{lij}. Unlike the case for compatible $\mathfrak{so}(n)$-valued test forms, the set of compatible $\mathrm{End}(TM)$-valued test forms is in general not a superset of the smooth $\mathrm{End}(TM)$-valued test forms, as discontinuities in f can force discontinuities in $\hat{\phi}$.

Note that if $\hat{\phi} \in \mathcal{A}(f, \mathcal{T}, M)$, then $\hat{\phi} \in \mathcal{A}(f \cdot h, \mathcal{T}, M)$ for any continuous piecewise-C^2 map $h : M \to O(n)$, so the dependence of $\mathcal{A}(f, \mathcal{T}, M)$ on the choice of frame is limited to dependence on its discontinuities. Additionally, the validity of equations (2.19) and (2.20) does not depend on the choice of tangent frames E_e and E_p, because if $\tilde{E}_e$ is another choice of orthonormal tangent frame for e, then $\tilde{E}^T_e = E^T_e \cdot \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix}$ for some orthogonal matrix $h \in O(n-1)$. Therefore, we have on $\Phi^{-1}(\mathring{e})$,

$$\begin{aligned}
[i^*_{\mathring{e}} \hat{\phi}^T]_{\tilde{E}^T_e} &= \mathrm{Ad}\left(\begin{bmatrix} h^{-1} & 0 \\ 0 & 1 \end{bmatrix}\right)\left([i^*_{\mathring{e}} \hat{\phi}^T]_{E^T_e}\right) = \mathrm{Ad}\left(\begin{bmatrix} h^{-1} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix}\right)\left([i^*_{\mathring{e}} \hat{\phi}^{T'}]_{E^{T'}_e}\right) \\
&= \mathrm{Ad}\left(\begin{bmatrix} h^{-1} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix}\right)\left([i^*_{\mathring{e}} \hat{\phi}^{T'}]_{\tilde{E}^{T'}_e}\right) \\
&= \mathrm{Ad}\left(\begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix}\right)\left([i^*_{\mathring{e}} \hat{\phi}^{T'}]_{\tilde{E}^{T'}_e}\right).
\end{aligned}$$

Similar reasoning shows that if an alternative (similarly oriented) tangential orthonormal frame $\tilde{E}_p$ is chosen for p, then equation (2.20) is valid for $[{\chi^T_{s,p}}^* \hat{\phi}^T]_{\tilde{E}^T_{p,e}}$ as well.

We may also define bijections $\Psi_{E^T_e} : C^0\Omega^k(e; \mathfrak{gl}(n)) \to C^0\Omega^k(e; \mathrm{End}(TM))$ and $\Psi_{E^T_{p,e}} : C^0\Omega^k(p; \mathfrak{gl}(n)) \to C^0\Omega^k(p; \mathrm{End}(TM))$ in exactly the same way as Ψ_f, but using the frames E^T_e and $E^T_{p,e}$ defined on e and p respectively.

We can then define the following endomorphism-valued forms, which are a 2-form defined on the codimension-0 polytope T, a pair of 1-forms defined on the codimension-1 polytope $e = T \cap T'$, and a 0-form defined on the codimension-2 polytope $p \subset T \cap e$

respectively:

$$\begin{aligned}
\hat{R} &:= \Psi_f(f^*\Omega), \\
\hat{\mathrm{I\!I}}_e^T &:= \Psi_{E_e^T}(\mathrm{I\!I}_e^T), \\
\hat{\mathrm{I\!I}}_e^{T'} &:= \Psi_{E_e^{T'}}(\mathrm{I\!I}_e^{T'}), \\
\hat{\Theta}_p^T &:= \Psi_{E_{p,e}^T}(\Theta_p w_n^{n-1}) = \Theta_p(-\vec{n} \otimes \vec{\nu}^\flat + \vec{\nu} \otimes \vec{n}^\flat).
\end{aligned}$$

Note that for consistency, when defining $\hat{\Theta}_p^T$, we must take care to use the face e such that $E_{p,e}^T$ has $-\vec{n}^T$ as its last vector, so that the rotation from $E_{p,e}^T$ to $E_{p,e'}^T$ (e' being the other codimension-1 face meeting p) is in the counterclockwise direction. Otherwise, as mentioned at the end of section 2.2.3, the sign of Θ_p will need to flip.

In coordinates, these forms can explicitly be written as

$$\begin{aligned}
\langle \hat{R} f_j, f_i \rangle_T &= {f^T}^* \Omega_j^i, \\
\langle \hat{\mathrm{I\!I}}_e{}^T (E_e^T)_j, (E_e^T)_i \rangle_T &= \begin{cases} 0 & \text{if } i, j < n, \\ \langle \nabla^T((E_e^T)_j), (E_e^T)_i \rangle_T & \text{if } i = n \text{ or } j = n, \end{cases} \\
\langle \hat{\Theta}_p^T (E_{p,e}^T)_j, (E_{p,e}^T)_i \rangle_T &= \begin{cases} \Theta_p & \text{if } i = n, j = n-1, \\ -\Theta_p & \text{if } i = n-1, j = n, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{2.21}$$

Here $\langle \cdot, \cdot \rangle_T$ and ∇^T refer to the inner product and covariant derivative induced by the metric g^T, respectively.

Frame-independent expressions for $\hat{R}$, $\hat{\mathrm{I\!I}}_e^T$, and $\hat{\Theta}_p^T$ are given in the lemma below.

Lemma 2. *On each $T \subseteq M$, $\hat{R}$ is the usual Riemann curvature tensor, i.e. it is the* $\mathrm{End}(TM)$*-valued 2-form given by*

$$\hat{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

On each $e \subset T$, $\hat{\mathrm{I\!I}}_e^T$ is given as follows. For all $x \in \mathring{e}$, $X, Y \in T_xM$, and $Z \in T_xe$,

$$\langle \hat{\mathrm{I\!I}}_e^T(Z)X, Y\rangle_T = \langle \nabla_Z^T \vec{n}, Y\rangle_T \langle X, \vec{n}\rangle_T - \langle \nabla_Z^T \vec{n}, X\rangle_T \langle Y, \vec{n}\rangle_T.$$

On each $p \subset T$, $\hat{\Theta}_p^T$ is the $\mathrm{End}(TM)$*-valued 0-form given by multiplying the angle defect Θ_p times the infinitesimal generator of counterclockwise rotation in the plane orthogonal to p in T.*

Proof. The claims about $\hat{R}$ and $\hat{\Theta}_p^T$ are clear from their definitions. To derive the formula for $\hat{\mathrm{I\!I}}_e^T$, let E_i be shorthand for $(E_e^T)_i$, and let $X = X^iE_i$, $Y = Y^iE_i$, and $Z = Z^iE_i$. Assume that Z is tangent to e, so that $Z^n = 0$. Then

$$\begin{aligned}
&\langle \nabla_Z^T \vec{n}, Y\rangle_T \langle X, \vec{n}\rangle_T - \langle \nabla_Z^T \vec{n}, X\rangle_T \langle Y, \vec{n}\rangle_T \\
&= Y^i \langle \nabla_Z^T E_n, E_i\rangle_T X^n - X^j \langle \nabla_Z^T E_n, E_j\rangle_T Y^n \\
&= X^n \langle \nabla_Z^T E_n, E_i\rangle_T Y^i + X^j \langle \nabla_Z^T E_j, E_n\rangle_T Y^n \\
&= \sum_{\substack{i,j=1 \\ i=n \text{ or } j=n}}^{n} X^j \langle \nabla_Z^T E_j, E_i\rangle_T Y^i.
\end{aligned}$$

Above, we used the fact that $\langle \nabla_Z^T E_n, E_j\rangle_T = -\langle \nabla_Z^T E_j, E_n\rangle_T$. We get the claimed result upon comparison with (2.21). $\square$

If f is any compatible frame and $\hat{\phi} \in \mathcal{A}(f, \mathcal{T}, M)$, then we define

$$\langle\langle \hat{R}_{\mathrm{dist}}, \hat{\phi}\rangle\rangle := \langle\langle f^*\Omega_{\mathrm{dist}}, \Psi_f^{-1}(\hat{\phi})\rangle\rangle.$$

The following theorem gives a more explicit formula for $\hat{R}_{\mathrm{dist}}$ and demonstrates that $\hat{R}_{\mathrm{dist}}$ is well-defined.

Theorem 2. *If $\hat{\phi} \in \mathcal{A}(f, \mathcal{T}, M)$ for some compatible frame f, then*

$$\langle\langle \hat{R}_{\mathrm{dist}}, \hat{\phi}\rangle\rangle = \sum_{T \subseteq M} \int_{\mathring{T}} \langle \hat{R} \wedge \hat{\phi}\rangle - \sum_{\mathring{e} \subset \mathring{M}} \int_{\mathring{e}} \left[\!\left[\langle \hat{\mathrm{I\!I}}_e \wedge i_{\mathring{e}}^* \hat{\phi}\rangle \right]\!\right] + \sum_{\mathring{p} \subset \mathring{M}} \int_{\mathring{p}} \langle \hat{\Theta}_p \wedge i_{\mathring{p}}^* \hat{\phi}\rangle. \tag{2.22}$$

Here $\left[\!\!\left[\langle \hat{\mathrm{I\!I}}_e \wedge i^*_{\mathring{e}}\hat{\phi}\rangle\right]\!\!\right]$ *is defined as* $\langle \hat{\mathrm{I\!I}}^T_e \wedge i^*_{\mathring{e}}\hat{\phi}|_T\rangle - \langle \hat{\mathrm{I\!I}}^{T'}_e \wedge i^*_{\mathring{e}}\hat{\phi}|_{T'}\rangle$*, which is interpreted as the jump in second fundamental form, while* $\langle \hat{\Theta}_p \wedge i^*_{\mathring{p}}\hat{\phi}\rangle$ *is defined as* $\langle \hat{\Theta}^T_p \wedge i^*_{\mathring{p}}\hat{\phi}|_T\rangle$*, where* T *is any of the* n*-dimensional polytopes containing* p *as a face.*

Proof. This follows fairly simply from the definitions (2.21) and the change of basis formula (2.18). Let $\phi = \Psi_f^{-1}(\hat{\phi})$, which is a compatible test form. Then:

$$\begin{aligned}
\langle \hat{R} \wedge \hat{\phi}\rangle &= \langle \Psi_f^{-1}(\hat{R}) \wedge \Psi_f^{-1}(\hat{\phi})\rangle = \langle f^{T*}\Omega \wedge \phi\rangle \\
\langle \hat{\mathrm{I\!I}}^T_e \wedge i^*_{\mathring{e}}\hat{\phi}|_T\rangle &= \langle \Psi^{-1}_{E^T_e}(\hat{\mathrm{I\!I}}^T_e) \wedge \Psi^{-1}_{E^T_e}(i^*_{\mathring{e}}\hat{\phi}|_T)\rangle = \langle \mathrm{I\!I}^T_e \wedge \mathrm{Ad}(\mu^T_e)(i^*_{\mathring{e}}\phi)\rangle \\
\langle \hat{\Theta}^T_p \wedge i^*_{\mathring{p}}\hat{\phi}|_T\rangle &= \langle \Psi^{-1}_{E^T_{p,e}}(\hat{\Theta}^T_p) \wedge \Psi^{-1}_{E^T_{p,e}}(i^*_{\mathring{p}}\hat{\phi}|_T)\rangle = \langle \Theta_p w_n^{n-1} \wedge \mathrm{Ad}(A^T_{p,e}\mu^T_e)(i^*_{\mathring{p}}\phi)\rangle.
\end{aligned}$$

From here, we can apply the identity $\langle \alpha \wedge \mathrm{Ad}(h)(\beta)\rangle = \langle \mathrm{Ad}(h^{-1})(\alpha) \wedge \beta\rangle$ which is valid for $\mathfrak{so}(n)$-valued forms. The only complication arises from the jump in second fundamental form. Note that because $\mu^{T'}_e = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \mu^T_e$, we have

$$\begin{aligned}
\langle \mathrm{Ad}((\mu^T_e)^{-1})(\mathrm{I\!I}^T_e) \wedge i^*_{\mathring{e}}\phi\rangle - &\langle \mathrm{Ad}((\mu^{T'}_e)^{-1})(\mathrm{I\!I}^{T'}_e) \wedge i^*_{\mathring{e}}\phi\rangle \\
&= \langle \mathrm{Ad}((\mu^T_e)^{-1})(\mathrm{I\!I}^T_e) - \mathrm{Ad}((\mu^T_e)^{-1})(-\mathrm{I\!I}^{T'}_e) \wedge i^*_{\mathring{e}}\phi\rangle \\
&= \langle \mathrm{Ad}((\mu^T_e)^{-1})([\![\mathrm{I\!I}_e]\!]) \wedge i^*_{\mathring{e}}\phi\rangle
\end{aligned}$$

□

Note that, since $i^*_{\mathring{p}}\hat{\phi}|_T$ is not single-valued due to (2.20), at first glance the last term seems to have some ambiguity. However, $\hat{\Theta}^T_p$ is invariant under rotations of the normal vectors $\vec{\nu}$ and $\vec{n}$ with which it is defined. More precisely, if $h = \exp\left(s\ r^T_{p,e}(x) w_n^{n-1}\right)$, then $[\hat{\Theta}^T_p]_{E^T_{p,e}} = [\hat{\Theta}^T_p]_{E^T_{p,e}\cdot h^{-1}}$. So, because $[{\chi^T_{s,p}}^*\hat{\phi}^T]_{E^T_{p,e}} = \mathrm{Ad}(h)([{\chi^T_{0,p}}^*\hat{\phi}^T]_{E^T_{p,e}})$ by (2.20) and

$\Psi_{E^T_{p,e}}$ is an isometry, we have

$$\begin{aligned}
\langle \hat{\Theta}^T_p \wedge {\chi^T_{s,p}}^* \hat{\phi}^T \rangle &= \langle [\hat{\Theta}^T_p]_{E^T_{p,e}} \wedge [{\chi^T_{s,p}}^* \hat{\phi}^T]_{E^T_{p,e}} \rangle \\
&= \langle [\hat{\Theta}^T_p]_{E^T_{p,e}} \wedge \mathrm{Ad}(h)([{\chi^T_{0,p}}^* \hat{\phi}^T]_{E^T_{p,e}}) \rangle \\
&= \langle \mathrm{Ad}(h^{-1})([\hat{\Theta}^T_p]_{E^T_{p,e}}) \wedge [{\chi^T_{0,p}}^* \hat{\phi}^T]_{E^T_{p,e}} \rangle \\
&= \langle [\hat{\Theta}^T_p]_{E^T_{p,e} \cdot h^{-1}} \wedge [{\chi^T_{0,p}}^* \hat{\phi}^T]_{E^T_{p,e}} \rangle \\
&= \langle [\hat{\Theta}^T_p]_{E^T_{p,e}} \wedge [{\chi^T_{0,p}}^* \hat{\phi}^T]_{E^T_{p,e}} \rangle \\
&= \langle \hat{\Theta}^T_p \wedge {\chi^T_{0,p}}^* \hat{\phi}^T \rangle.
\end{aligned}$$

So, in (2.22), "$i^*_{\mathring{p}} \hat{\phi}|_T$" could be thought of as a shorthand for ${\chi^T_{s,p}}^* \hat{\phi}^T$, where s may be any convenient number in $[0, 1]$. Concretely, this means each $x \in \mathring{p}$ can be approached along a ray of any convenient angle, and the result will be the same. Additionally, due to equation (2.15), this term does not depend on the choice of T of which p is a face.

The right side of equation (2.22) does not depend on f at all, so as long as $\hat{\phi} \in \mathcal{A}(f, \mathcal{T}, M)$ for some compatible frame f, this definition can be used to compute the distributional curvature. The caveats on the integrals are the same as for (2.16): the integrals over $\mathring{e}$ do not depend on the choice of T, but care must be chosen so that the integral is evaluated with the orientation induced on e by the orientation of T, and the integrals over $\mathring{p}$ must be evaluated using the orientation induced from the face e such that $E^T_{p,e}$ has an inward-pointing normal vector as its last entry.

This expression has some close similarities with the *densitized distributional curvature* investigated in [32]. In fact, the two expressions are equivalent, with the main difference being the choice of how to represent the second fundamental form and which indices are raised/lowered. See Appendix 5.2 for a proof. As explained in [32], various traces of this distribution can be taken to obtain the Ricci curvature, Einstein tensor, and the scalar curvature.

2.3 Construction of Compatible Frames

Most of Section 2.2 would be meaningless if a compatible frame did not exist. In this section, we will give a proof of the following theorem:

Theorem 3. *Let g be a Regge metric for the mesh $\mathcal{T}$ such that each polytope $T \subseteq M$ has a blow-up which is a closed convex polytope in $\mathbb{R}^n$, and suppose there exists a C^2 homotopy of Regge metrics $g(t)$ such that $g(1) = g$, $g(0) =: g_0$ is a smooth metric, and there exists a smooth g_0-orthonormal frame f_0. Then there exists a C^2 homotopy of frames $f(t)$ such that $f(0) = f_0$, $f^T(t)$ is $g(t)$-orthonormal when restricted to each T, $f(t)$ satisfies conditions 1, 2, and 3 of compatible frames, and the maps $(t, x) \mapsto F^T(t)(x)$ vary continuously as maps $[0, 1] \times B_T \to \mathcal{F}_{GL}(T)$, where F^T is the blown-up frame from condition 1.*

The general proof strategy is as follows: For each codimension-2 polytope p, we evolve the frame E_p into a $g(t)$-orthonormal frame $E_p(t)$. Then, for each codimension-1 polytope $e \supset p$, we produce a frame $E_e(t)$ whose associated matrix $A_{p,e}$ is constant in time. Then, on each codimension-0 polytope T, we use Lemma 4 (appearing below) to produce homotopies of orthonormal frames $F^T(t) : B_T \to \mathcal{F}_O(T)$ having the same matrices μ_e^T as f_0 does, and $r_{p,e}^T(t) = \theta_p^T(t) - \theta_p^T(0)$. The frames $f^T(t) = F^T(t) \circ \Phi^{T^{-1}}$ are then compatible by construction.

First, we need a result that allows us to extend frames. This is a technical result that ultimately relies on the fact that functions can be extended smoothly from closed sets. We will specifically use the theorem as stated in [60] (see also [41, Lemma 2.26]), which can be simplified for our purposes in the way detailed below. In the remainder of this section, we will stop abusing terminology and use the word "polytope" to refer to a genuine polytope (as opposed to the image of a polytope under a smooth embedding). Our reason for doing so is that the extension procedure that we will soon describe takes place on the blow-up B_T, which we will assume to be a polytope. (In fact we will assume it to be convex.) We will use the notation $\hat{T}$ for polytopes below; ultimately we will take $\hat{T} = B_T$ when we begin constructing a compatible frame.

Theorem 4 ([60])**.** *Let $\hat{T} \subset \mathbb{R}^n$ be a closed, convex polytope. If $a : \hat{T} \to \mathbb{R}$ is a C^r function, $0 \leq r \leq \infty$, then there exists a C^r extension $A : \mathbb{R}^n \to \mathbb{R}$ such that $A|_{\hat{T}} = a$.*

We will use this as the base of a lemma allowing us to extend partially defined functions from the boundary of a polytope to the interior.

Lemma 3. *Let $\hat{T} \subset \mathbb{R}^n$ be a closed, convex n-dimensional polytope. Denote by $\{\hat{e}_i\}_{i=1}^N$ a set of codimension-1 faces of $\hat{T}$, and let $\hat{p}_{ij} := \hat{e}_i \cap \hat{e}_j$. Then for any collection of C^r ($r \geq 1$) functions $a^i : \hat{e}_i \to \mathbb{R}$ such that $a^i|_{\hat{p}_{ij}} = a^j|_{\hat{p}_{ij}}$ for all i, j, there exists a Lipschitz continuous function $A : \hat{T} \to \mathbb{R}$ such that $A|_{\hat{e}_i} = a^i$ for each i and $A|_{\hat{T} \setminus \bigcup_{i,j} \hat{p}_{ij}}$ is a C^r function.*

Note that if $\hat{T}$ happens to be a manifold with corners, the Whitney extension theorem can be used directly to produce an extension A which is globally C^r.

Proof. Per Theorem 4, each a^i can be extended to a C^r function $A^i : \mathbb{R}^n \to \mathbb{R}$. Since $\hat{e}_i$ is a convex polytope of dimension $n-1$, it lies completely in a hyperplane $E_i \subset \mathbb{R}^n$. Set $\lambda^i(x) := \pm\text{dist}(x, E_i)$, where the sign is positive if x is on the same side of E_i as $\hat{T}$ and negative otherwise. This makes it an affine function.

Now let $\hat{\Lambda}_i(x) = \prod_{j \neq i} \lambda^j(x)$ and $\hat{\Lambda}_{ij}(x) = \prod_{k \notin \{i,j\}} \lambda^k(x)$. The extension we seek is

$$A(x) = \frac{\sum_i A^i(x)\hat{\Lambda}_i(x)}{\sum_i \hat{\Lambda}_i(x)}.$$

This function satisfies $A|_{\hat{e}_i} = a^i$, and A is C^r on the set such that all of the λ^i's are non-negative and no more than one of the λ^i's is equal to zero, which is exactly the set $\hat{T} \setminus \bigcup_{i,j} \hat{p}_{ij}$.

All that remains is to prove that the first partial derivatives of A are bounded if $a^i|_{\hat{p}_{ij}} =$

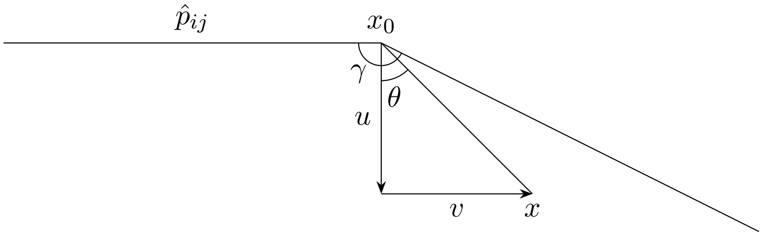

Figure 2.1: A diagram of a possible configuration between x, a point on the interior of $\hat{T}$, and x_0, the nearest point in $\hat{p}_{ij}$ to x. The diagram pictured can be imagined as lying in the intersection of $\hat{T}$ with the plane containing x, x_0, and $x_0 + u$.

$a^j|_{\hat{p}_{ij}}$ for all i, j. To do this, we compute (abandoning the Einstein summation convention)

$$\begin{aligned} dA &= \frac{\sum_i A^i \sum_{j\neq i} \hat{\Lambda}_{ij} d\lambda^j \left(\sum_i \hat{\Lambda}_i\right) - \left(\sum_i A^i \hat{\Lambda}_i\right)\left(\sum_i \sum_{j\neq i} \hat{\Lambda}_{ij} d\lambda^j\right)}{\left(\sum_i \hat{\Lambda}_i\right)^2} + \frac{\sum_i \hat{\Lambda}_i dA^i}{\sum_i \hat{\Lambda}_i} \\ &= \frac{\sum_{i,j} \sum_{l\neq i} (A^i - A^j) \hat{\Lambda}_j \hat{\Lambda}_{il} d\lambda^l}{\left(\sum_i \hat{\Lambda}_i\right)^2} + \frac{\sum_i \hat{\Lambda}_i dA^i}{\sum_i \hat{\Lambda}_i}. \end{aligned}$$

Next note that for any $x \in \hat{T}$, and any pair of indices i, j, there exists a nearest point $x_0 \in \hat{p}_{ij}$ to x. The difference $x - x_0$ has a component $\vec{u}$ which is normal to $T\hat{p}_{ij}$ and a tangential component $\vec{v}$. Because $\hat{p}_{ij}$ is a closed submanifold, the closest point x_0 has the property that either $\vec{v} = 0$ or x_0 lies on the boundary of $\hat{p}_{ij}$. In the latter case the angle between $\vec{u}$ and $x - x_0$ is less than $\gamma - \frac{\pi}{2}$, where $\gamma < \pi$ is the largest interior angle of the polytope $\hat{T}$, and thus $\|\vec{v}\| \leq \tan(\gamma - \frac{\pi}{2})\|\vec{u}\|$ (see Figure 3.1 for a diagram).

In both cases, $\|x - x_0\|$ is bounded by a constant multiple of $\|\vec{u}\|$, which is in turn bounded by a constant multiple of $|\langle \vec{n}_i, \vec{u}\rangle| + |\langle \vec{n}_j, \vec{u}\rangle|$, since $(\vec{n}_i, \vec{n}_j)$ are linearly independent and normal to $T\hat{p}_{ij}$. $\lambda^i(x)$ is exactly equal to $-\langle \vec{n}_i, \vec{u}\rangle$ since $x_0 \in e_i$, and similarly for λ^j. Therefore there is a constant C such that $\|x - x_0\| \leq C(\lambda^i(x) + \lambda^j(x))$.

By the Taylor theorem, since $A^i(x_0) = A^j(x_0)$ and A^i, A^j are both C^r and bounded on $\hat{T}$, this implies there exists a constant M such that $|A^i(x) - A^j(x)| \leq M(\lambda^i(x) + \lambda^j(x))$.

We can use this, together with the fact that $\|dA^i\|$ is bounded above by some constant M' on $\hat{T}$ and $\|d\lambda^l\| = 1$, to bound $\|dA\|$:

$$\begin{aligned}
\|dA\| &\leq \frac{\sum_{i,j}\sum_{l\neq i}|A^i - A^j|\hat{\Lambda}_j\hat{\Lambda}_{il}\|d\lambda^l\|}{\left(\sum_j \hat{\Lambda}_j\right)^2} + \frac{\sum_i \hat{\Lambda}_i\|dA^i\|}{\sum_j \hat{\Lambda}_j} \\
&\leq M\frac{\sum_{i,j}\sum_{l\neq i}(\lambda^i + \lambda^j)\hat{\Lambda}_j\hat{\Lambda}_{il}}{\left(\sum_j \hat{\Lambda}_j\right)^2} + M'\frac{\sum_i \hat{\Lambda}_i}{\sum_j \hat{\Lambda}_j}.
\end{aligned}$$

Now we can use the fact that $\hat{\Lambda}_j\hat{\Lambda}_{il}(\lambda^i + \lambda^j) = \hat{\Lambda}_l(\hat{\Lambda}_i + \hat{\Lambda}_j)$, so

$$\|dA\| \leq M\frac{\sum_{i,j}\sum_{l\neq i}\hat{\Lambda}_l(\hat{\Lambda}_i + \hat{\Lambda}_j)}{\left(\sum_j \hat{\Lambda}_j\right)^2} + M' \leq 2MN + M'$$

This gives an upper bound for $\|dA\|$ on the interior of $\hat{T}$, so A is a Lipschitz function. □

We will use the above lemma to extend evolving frames defined on faces of a convex polytope $\hat{T}$ to the interior of the polytope. In the statement of the next lemma, the pulled back frame bundle $\Phi^*\mathcal{F}_{GL}(T)$ is the bundle over $\hat{T}$ such that the fiber over $x \in \hat{T}$ is the set of bases for $T_{\Phi(x)}(T)$. Furthermore, $S_d(\hat{T})$ refers to the codimension-d stratum of $\hat{T}$, which is the union of the relative interiors of faces of $\hat{T}$ that have codimension d.

Lemma 4. *Let $\hat{T} \subset \mathbb{R}^n$ be a closed convex polytope of dimension n and $\Phi : \hat{T} \to M$ a continuous embedding that is smooth on $\hat{T}\setminus\bigcup_{d\geq 2} S_d(\hat{T})$. We will refer to the image of Φ by T. Let $g(t)$ be a C^2 (in both space and time) nondegenerate symmetric bilinear form on T for $t \in [0,1]$, and let $g_0 = g(0)$. Suppose that the following are true:*

1. *There are C^2 (in both space and time) frames $f(t)|_{\hat{e}_i} : \hat{e}_i \to \Phi^*\mathcal{F}_{GL}(T)$, defined on some subset $\{\hat{e}_i\}_{i=1}^N$ of the codimension-1 faces of $\hat{T}$.*

2. *For each $i, j \in 1, \ldots, N$, $f(t)|_{\hat{e}_i}(x) = f(t)|_{\hat{e}_j}(x)$ for all $t \in [0,1]$ and $x \in \hat{p}_{ij} = \hat{e}_i \cap \hat{e}_j$, and $\frac{d}{dt}[g(t)((f(t)|_{\hat{e}_i})_j, (f(t)|_{\hat{e}_i})_k)] = 0$ for all i, j, k, t.*

3. There is a smooth g_0-orthonormal frame $f_0 : \hat{T} \to \Phi^*\mathcal{F}_{GL}(T)$ such that $f_0(x) = f(0)|_{\hat{e}_i}(x)$ for all i and all $x \in \hat{e}_i$.

Then there is a C^2 homotopy of frames $f(t) : \hat{T} \to \Phi^*\mathcal{F}_{GL}(T)$ which is C^2 on $\hat{T} \setminus \bigcup_{i,j} \hat{p}_{ij}$ such that $f(t)(x) = f(t)|_{\hat{e}_i}(x)$ for all i and all $x \in \hat{e}_i$, $f(0)(x) = f_0(x)$ for all $x \in \hat{T}$, $\frac{d}{dt}[g(t)(f(t)_j, f(t)_k)] = 0$ for all j, k, t, and $f(t)$ is Lipschitz continuous for each t.

Proof. The proof will proceed as follows. We derive an ordinary differential equation for the change of basis $u : [0,1] \times \hat{T} \to GL(n)$ so that $f_0 \cdot u(t)$ is $g(t)$-orthonormal, show that it has solutions for all time, and design the free parameters of the equation so that the solution is C^2 and Lipschitz in space and agrees with $f(t)|_{\hat{e}_i}$.

Suppose, first, that a frame $f(t)$ satisfies $\frac{d}{dt}[g(t)(f(t)_j, f(t)_k)] = 0$ for all j, k. Then let $u(t) : \hat{T} \setminus \bigcup_{i,j} \hat{p}_{ij} \to GL(n)$ be the unique matrix such that $f(t) = f(0) \cdot u(t)$. The condition we have placed on $f(t)$ will allow us to find an ordinary differential equation for u. Let $\tilde{g}(t)_{ij} := g(t)(f(t)_i, f(t)_j)$, and $\tilde{\sigma}(t)_{ij} := \dot{g}(t)(f(0)_i, f(0)_j)$, both symmetric matrices. Then expanding out the inner products, we get

$$\begin{aligned}
\frac{\partial \tilde{g}(t)_{ij}}{\partial t} &= \frac{d}{dt}[u_i^k u_j^l g(t)(f(0)_k, f(0)_l)] \\
&= \dot{u}_i^k u_j^l g(t)(f(0)_k, f(0)_l) + u_i^k \dot{u}_j^l g(t)(f(0)_k, f(0)_l) \\
&\qquad + u_i^k u_j^l \dot{g}(t)(f(0)_k, f(0)_l) \\
&= (u^{-1})_k^m \dot{u}_i^k g(t)(f(t)_m, f(t)_j) + (u^{-1})_l^m \dot{u}_j^l g(t)(f(t)_i, f(t)_m) \\
&\qquad + u_i^k u_j^l \tilde{\sigma}(t)_{kl} \\
&= (u^{-1}\dot{u})_i^m \tilde{g}_{mj} + (u^{-1}\dot{u})_j^m \tilde{g}_{mi} + (u^\intercal \tilde{\sigma}(t) u)_{ij} \\
&= [\tilde{g}u^{-1}\dot{u} + (\tilde{g}u^{-1}\dot{u})^\intercal + u^\intercal \tilde{\sigma}(t) u]_{ij} \\
&= [(\tilde{g}u^{-1}\dot{u} + \frac{1}{2}u^\intercal \tilde{\sigma}(t) u) + (\tilde{g}u^{-1}\dot{u} + \frac{1}{2}u^\intercal \tilde{\sigma}(t) u)^\intercal]_{ij}.
\end{aligned}$$

This shows that the condition $\frac{\partial \tilde{g}}{\partial t} = 0$ is equivalent to the condition that $\tilde{g}u^{-1}\dot{u} + \frac{1}{2}u^\intercal \tilde{\sigma}(t) u$ is a skew-symmetric matrix, which we will call K. Put another way, for any skew-symmetric matrix-valued map $K : [0,1] \times \hat{T} \to \mathfrak{so}(n)$, the equation

$$\dot{u} = u\tilde{g}^{-1}(K - \frac{1}{2}u^{\intercal}\tilde{\sigma}(t)u) \tag{2.23}$$

holds if and only if $\frac{\partial \tilde{g}}{\partial t} = 0$. Assuming $f(0)$ is $g(0)$-orthonormal, this is equivalent to saying that $f(t)$ is $g(t)$-orthonormal for all t, and hence $\tilde{g} = \eta = \operatorname{diag}(1, \dots, 1, -1, \dots, -1)$ with the number of negative elements equal to some number k and the number of positive elements equal to $n - k$.

In the next few paragraphs, we will argue that (2.23) admits a unique solution $u : [0, 1] \times \hat{T} \to GL(n)$ for any Lipschitz choice of K. We start by constructing a solution for one particular choice of K, and then we leverage it to construct solutions for other choices of K.

Let $\tilde{G}(t)_{ij} := g(t)(f_{0i}, f_{0j})$. If $X(t)$ is the LDL square root of $\tilde{G}(t)^{-1}$, so $\tilde{G}(t)^{-1} = X^T \eta X$, then $f_0 \cdot X(t)$ is a $g(t)$-orthonormal frame which is C^2 in space and time. Thus, X satisfies equation (2.23) with some matrix-valued function $K' : [0, 1] \times \hat{T} \to \mathfrak{so}(n)$.

Now consider the differential equation

$$\dot{V} = V\eta K - \eta K' V. \tag{2.24}$$

We claim that if $V : [0, 1] \times \hat{T} \to O(n - k, k)$ satisfies (2.24), then $u = XV$ satisfies (2.23). This is straightforward to verify:

$$\begin{aligned}
\dot{u} &= \dot{X}V + X\dot{V} \\
&= X\eta(K' - \frac{1}{2}X^{\intercal}\tilde{\sigma}X)V + X(V\eta K - \eta K'V) \\
&= X\eta K'V - \frac{1}{2}(uV^{-1})\eta(uV^{-1})^{\intercal}\tilde{\sigma}u + u\eta K - X\eta K'V \\
&= u\eta K - \frac{1}{2}uV^{-1}\eta V^{-\intercal}u^{\intercal}\tilde{\sigma}u \\
&= u\eta(K - \frac{1}{2}V^{\intercal}V^{-\intercal}u^{\intercal}\tilde{\sigma}u) \\
&= u\eta(K - \frac{1}{2}u^{\intercal}\tilde{\sigma}u).
\end{aligned}$$

Note that for fixed $x \in \hat{T}$, the right-hand side of (2.24) is the time-dependent vector field $W(V, t) = V\eta K(t) - \eta K'(t)V$ on the manifold $O(n - k, k)$, and at each time it is also linear as a map $M^{n\times n}(\mathbb{R}) \to M^{n\times n}(\mathbb{R})$, so if $K, K' : [0,1]\times \hat{T} \to \mathfrak{so}(n)$ are Lipschitz continuous in time then there exists a unique solution $V : [0,1]\times\hat{T} \to O(n-k,k)$ satisfying (2.24). If, in addition, K and K' are Lipschitz, C^2 in space, and C^1 in time when restricted to $[0,1]\times(\hat{T}\backslash\bigcup_{i,j}\hat{p}_{ij})$, then V is C^2 and Lipschitz in both space and time when restricted to the same set. We already know that K' satisfies all these conditions (because X is in the same differentiability class as G), so we just need to choose K appropriately.

Since we have C^2 frames $f(t)|_{\hat{e}_i}$ on the faces $\hat{e}_i$, which are single-valued at $\hat{p}_{ij}$, and $\frac{dg(t)((f(t)|_{\hat{e}_i})_j,(f(t)|_{\hat{e}_i})_k)}{dt} = 0$, there are C^2 matrix-valued maps $u^i : [0,1]\times\hat{e}_i \to GL(n)$ and $K^i : [0,1]\times\hat{e}_i \to \mathfrak{so}(n)$ which are single-valued on $\hat{p}_{ij}$ and satisfy $\eta {u^i}^{-1}\dot{u}^i + \frac{1}{2}{u^i}^{\intercal}\tilde{\sigma}(t)u^i = K^i$. By Lemma 3, the K^i's can be extended (by extending each coordinate) to a continuous map $K : [0,1]\times\hat{T}\to\mathfrak{so}(n)$ which is C^2 and has bounded first derivatives on $[0,1]\times(\hat{T}\backslash\bigcup_{i,j}\hat{p}_{ij})$.

This is sufficient to assert the existence of a unique family of maps $u(t) : \hat{T}\to GL(n)$ satisfying $\dot{u} = u\eta(K - \frac{1}{2}u^{\intercal}\tilde{\sigma}(t)u)$ for $t\in[0,1]$ and $x\in\hat{T}$, $u(0) = I$, and $u(t)|_{\hat{e}_i} = u^i(t)$ (since the solution to this ordinary differential equation is unique at each $x\in\hat{e}_i$). By the smooth dependence on parameters, u also has continuous second spatial derivatives when restricted to $[0,1]\times\hat{T}\backslash\bigcup_{i,j}\hat{p}_{ij}$ and is Lipschitz on $[0,1]\times\hat{T}$. □

Corollary 1. *Let T, $g(t)$, and f_0 be as above. Then there exists a C^2 homotopy of C^2 frames $f(t) : T\to\mathcal{F}_{GL}(T)$ such that $f(0) = f_0$ and $\frac{\partial}{\partial t}g(t)(f_i(t), f_j(t)) = 0$.*

Proof. In the proof of Lemma 4, $\hat{T}$ can be set equal to T and Φ can simply be the identity map. We no longer have any boundary conditions on $f(t)|_{\hat{e}_i}$. Then choosing a C^2 map $K : [0,1]\times T\to\mathfrak{so}(n)$ is enough to produce a solvable o.d.e. for u. $K = 0$ is a valid choice. □

Proof of Theorem 3. Firstly, for each $p\subset M$, pick a frame E_{p0} for Tp which is g_0-orthonormal and let $E_p(t) = (\tau_1,\ldots,\tau_{n-2})$ be a C^2 homotopy of $g(t)$-orthonormal frames on p which are C^2 in space and such that $E_p(0) = E_{p0}$. These frames could be found by applying Corollary 1. As usual, extra $g(t)$-orthonormal vectors can be appended to $E_p(t)$ to pro-

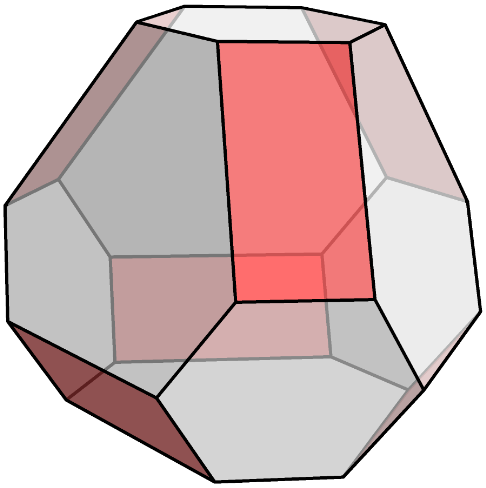

Figure 2.2: Blow-up of a solid tetrahedron. The regions shaded red are the sets $\overline{\Phi^{T^{-1}}(\mathring{p})}$ for the codimension-2 faces $p \subset T$. The frames $F^T(t)|_p$ are defined on these regions, and the frames $\tilde{E}_e(t)|_p$ are defined on the long sides of these regions.

duce frames $E_{p,e}(t) = (\tau_1, \ldots, \tau_{n-2}, \vec{\nu})$ and $E^T_{p,e}(t) = (\tau_1, \ldots, \tau_{n-2}, \vec{\nu}, \vec{n})$ which are in the same differentiability class and satisfy $E_{p,e}(0) = E_{p,e0}$ $E^T_{p,e}(0) = E^T_{p,e0}$ for each $p \subset e \subset T$. Also let E_{e0} be a g_0-orthonormal frame on e and E^T_{e0} be the same frame with the outward unit normal appended, and let $A_{p,e} : p \to O(n-1)$ be the map such that $E_{e0} = E_{p,e0} \cdot A_{p,e}$, and likewise let $A^T_{p,e} : p \to O(n)$ be the map such that $E^T_{e0} = E^T_{p,e0} \cdot A^T_{p,e}$.

For each codimension-1 face e and codimension-2 face $p \subset e$, let $\tilde{E}_e(t)|_p : \overline{\Phi^{T^{-1}}(\mathring{p})} \cap \overline{\Phi^{T^{-1}}(\mathring{e})} \to \mathcal{F}_O(e)$ be defined by the relation $\tilde{E}_e(t)|_p(x) = E_{p,e}(t)(\Phi^T(x)) \cdot A_{p,e}(\Phi^T(x))$, and let $\tilde{E}_{e0} : \overline{\Phi^{T^{-1}}(\mathring{e})} \to \mathcal{F}_O(e)$ be defined by $\tilde{E}_{e0}(x) = E_{e0}(\Phi^T(x))$. Since the sets $\overline{\Phi^{T^{-1}}(\mathring{p})} \cap \overline{\Phi^{T^{-1}}(\mathring{e})}$ for differing p are all disjoint and $\tilde{E}_e(0)|_p = \tilde{E}_{e0}$, by Lemma 4, there exists a smooth map $\tilde{E}_e(t) : \overline{\Phi^{T^{-1}}(\mathring{e})} \to \mathcal{F}_O(e)$ which extends $\tilde{E}_e(t)|_p$ for each $p \subset e$ and such that $\tilde{E}_e(0) = \tilde{E}_{e0}$. The outwards normal vector can be appended to obtain $E^T_e(t)$.

Define $F^T(t)|_e : \overline{\Phi^{T^{-1}}(\mathring{e})} \to \mathcal{F}_O(T)$ by $F^T(t)|_e(x) := \tilde{E}^T_e(t)(x) \cdot \mu^T_e(\Phi^T(x))$. Note that

$F^T(0)|_e(x) = \Phi^{T^*} f_0(x)$ for all $x \in \overline{\Phi^{T^{-1}}(\mathring{e})}$.

Next it is time to define $F^T(t)|_p$ for codimension-2 faces p of T. It needs to agree with $F^T(t)|_e$ wherever both frames are defined. Due to the fact that we are keeping μ_e^T and $A_{p,e}^T$ constant and we want $F^T(t)$ to be compatible, the only question is what the function $r_{p,e}^T(t) : p \to \mathbb{R}$ should be. Note that equation (2.13) is equivalent to

$$E_{p,e_j}^{T_j}(t) \cdot \exp\left(r_{p,e_j}^{T_j}(t) w_n^{n-1}\right) A_{p,e_j}^{T_j} \mu_{e_j}^{T_j} = E_{p,e_{j+1}}^{T_j}(t) \cdot A_{p,e_{j+1}}^{T_j} \mu_{e_{j+1}}^{T_j}.$$

In other words, it is equivalent to $F^T(t)|_e$ being equal to $F^T(t)|_p$ wherever both are defined. Since the matrices μ_e^T and $A_{p,e}^T$ are held fixed for the entire evolution, we require $r_{p,e_j}^{T_j}(t) - \theta_p^{T_j}(t)$ to be constant. Since $r_{p,e_j}^{T_j}(0) = 0$, this means $r_{p,e_j}^{T_j}(t) = \theta_p^{T_j}(t) - \theta_p^{T_j}(0)$.

Now we bring it all together. $F^T(t)|_p$ and $F^T(t)|_e$ are defined and C^2 on faces of the polytope $B_T \subseteq \mathbb{R}^n$ and single-valued everywhere, therefore they can be extended, for each t, to a section $F^T(t) : B_T \to \Phi^{T^*}\mathcal{F}_O(T)$ which remains orthonormal for all t, is C^2 on $B_T \backslash E_{B_T}$, and has bounded derivatives on $\mathring{B}_T$. This frame is C^2 in time and satisfies $F^T(0) = \Phi^{T^*} f_0$. Lastly, set $f^T(t)(x) = F^T(t)(\Phi^{-1}(x))$ for $x \in S_0(T) \cup S_1(T)$. The resulting frame satisfies properties 1-3 of a $g(t)$-compatible frame by construction, and $f^T(0) = f_0$, so $f(1)$ is a compatible frame. $\square$

One feature of this proof is that it suggests that the only obstruction for a frame satisfying conditions 1-3 of a compatible frame to also satisfy the fourth condition is the existence of a homotopy $g(t)$ between g and a continuous metric g_0 such that $\theta_p^{T_j}(0) = \theta_p^{T_j}(1) - r_{p,e_j}^{T_j}(1)$ for all p, j.

2.4 Generalized Gauss-Bonnet Theorem

In this section we will work out what happens when we remove the restriction that $i_{\partial M}^* \phi = 0$, in the case that $n = 2$ and M is a manifold with corners rather than a general polyhedral manifold. In this context can derive a generalization of the Gauss-Bonnet theorem.

When specialized to 2 dimensions, the expression (2.16) simplifies considerably, since $\mathfrak{so}(2)$ and $\Lambda^2(M)$ are both one-dimensional. In this case there exists a function $K^T : T \to \mathbb{R}$ such that $\Omega^T = K^T w_1^2 \otimes dA^T$, where dA^T is the positively oriented volume form on T induced by the metric g^T. K^T is precisely the Gauss curvature, and does not depend on f^T. Additionally, the adjoint action is trivial on $\mathfrak{so}(2)$, and if h is a smooth $SO(2)$-valued function which is a rotation by the angle θ, then $h^{-1}dh$ is equal to $\begin{bmatrix} 0 & -d\theta \\ d\theta & 0 \end{bmatrix} = w_2^1 \otimes d\theta$.

To make use of these simplifications, let us evaluate $\langle\langle f^*\Omega_{\text{dist}}, \frac{1}{2}\phi w_1^2\rangle\rangle$ when ϕ is an arbitrary smooth function on M. All of the steps in Section 2.2.2) are the same up until Equation (2.10), where we need to add some boundary terms. Without loss of generality we can assume that f^T is always positively oriented and, for all $e \subset \partial M$, $E_e^T = (\tau_e, \vec{n}_e)$ where $\vec{n}_e$ is the outward-pointing normal vector to e and τ_e is the unit tangent vector that makes the frame positively oriented. Therefore the matrices μ_e^T, when $e \subset \partial M$, are just rotations by some angle $\bar{\mu}_e$.

The first boundary terms come from the integrals along codimension-1 edges $e \subset \partial M$, and they are equal to

$$\begin{aligned} -\sum_{e\subset\partial M}\int_{\mathring{e}}\frac{1}{2}\langle f^*\omega\wedge\phi w_1^2\rangle &= -\frac{1}{2}\sum_{e\subset\partial M}\int_{\mathring{e}}\langle\big(\mu_e^{T-1}d\mu_e^T + \mathrm{Ad}(\mu_e^{T-1})(\mathrm{II}_e^T)\big)\wedge\phi w_1^2\rangle \\ &= -\frac{1}{2}\sum_{e\subset\partial M}\int_{\mathring{e}}\phi\langle\big(w_2^1\otimes d\bar{\mu}_e + \mathrm{II}_e^T\big)\wedge w_1^2\rangle \\ &= -\sum_{e\subset\partial M}\int_{\mathring{e}}\phi(d\bar{\mu}_e + (\mathrm{II}_e^T)_1^2). \end{aligned} \tag{2.25}$$

The other boundary terms come from the integrals along codimension-2 points $p \in \partial M$, and following equation (2.12), they can simply be expressed as

$$\begin{aligned} -\sum_{p\in\partial M}\frac{1}{2}\langle\sum_{T\ni p}\mathrm{Ad}((A_{p,e}^T\mu_e^T)^{-1})(r_{p,e}^T w_2^1)\wedge\phi w_1^2\rangle &= -\frac{1}{2}\sum_{p\in\partial M}\phi(p)\langle\sum_{T\ni p}r_{p,e}^T w_2^1, w_1^2\rangle \\ &= -\sum_{p\in\partial M}\phi(p)\sum_{T\ni p}r_{p,e}^T. \end{aligned}$$

We will now assume that the frame f was constructed as in Theorem 3, meaning the homotopy of frames is constructed such that the matrices $\mu_e^T(t)$ are constant throughout the evolution. We will also assume that a counterclockwise enumeration of the triangles $T_1, \dots, T_k$ and edges $e_1, \dots, e_{k+1}$ incident to $p \in \partial M$ is chosen much like in 2.2.3, with the notable difference that e_1 and e_{k+1} are only tangent to the triangles T_1 and T_k respectively.

One of the key facts for the proof of Theorem 3 is that $r_{p,e_j}^{T_j}(t) = \theta_p^{T_j}(t) - \theta_p^{T_j}(0)$. However, since $f(0)$ is continuous, $\sum_{j=1}^k \theta_p^{T_j}(0)$ is equal to the jump in angle between $f(0)$ and the two frames $(\tau_{e_{k+1}}, \vec{n}_{e_{k+1}})$ and $(-\tau_{e_1}, -\vec{n}_{e_1})$. In other words, $\sum_{j=1}^k \theta_p^{T_j}(0) = 2\pi m_p + \pi + \bar{\mu}_{e_{k+1}}(0) - \bar{\mu}_{e_1}(0)$ for some integer m_p. The $2\pi m_p$ term is necessary because $\bar{\mu}_{e_j}(0)$ is only well defined up to addition by 2π. The quantity $\bar{\mu}_{e_{k+1}} - \bar{\mu}_{e_1} + 2\pi m_p$ will from now on be shortened to $[\![\mu]\!]|_p$. Since these angles are kept fixed through the whole evolution, we get $\sum_{j=1}^k r_{p,e_j}^{T_j} = -[\![\mu]\!]|_p - \pi + \sum_{j=1}^k \theta_p^{T_j}$. So the additional angle defect terms take the form

$$- \sum_{p \in \partial M} \phi(p) \sum_{T \ni p} r_{p,e}^T = \sum_{p \in \partial M} \phi(p) \left([\![\mu]\!]|_p + \pi - \sum_{T \ni p} \theta_p^T \right). \tag{2.26}$$

Synthesizing (2.25) and (2.26) into the expression for $\langle\langle f^*\Omega, \frac{1}{2}\phi w_1^2 \rangle\rangle$, we can define the *distributional Gauss curvature*:

$$\begin{aligned} \langle\langle K_{\text{dist}}, \phi \rangle\rangle &:= \langle\langle f^*\Omega_{\text{dist}}, \frac{1}{2}\phi w_1^2 \rangle\rangle + \sum_{e \subset \partial M} \int_{\mathring{e}} \phi d\bar{\mu}_e - \sum_{p \in \partial M} \phi(p) [\![\mu]\!]|_p \qquad (2.27) \\ &= \sum_{T \subseteq M} \int_{\mathring{T}} K^T \phi dA - \sum_{\mathring{e} \subset \mathring{M}} \int_{\mathring{e}} \phi [\![\mathrm{I\!I}_e]\!]_1^2 + \sum_{p \in \mathring{M}} \Theta_p \phi(p) \\ &\quad - \sum_{e \subset \partial M} \int_{\mathring{e}} \phi \mathrm{I\!I}_{e\,1}^{\;2} + \sum_{p \in \partial M} \left(\pi - \sum_{T \ni p} \theta_p^T \right) \phi(p). \end{aligned}$$

This quantity is frame-independent and identical to the *densitized distributional Gauss curvature* investigated in [6, 27]. The fact that the frame dependence of the distributional curvature is concentrated in boundary terms is reminiscent of the following formula that is valid for a smooth metric and smooth frame, obtained from integration by parts:

$$\int_M f^*\omega_2^1 \wedge d\phi = \int_{\mathring{M}} \phi K dA - \int_{\partial M \setminus E_M} \phi f^* \omega_2^1.$$

Therefore we can interpret the boundary components of the distributional curvature as being a distributional version of the connection one-form pulled back to the boundary. The analogue of $\int_{\partial M \setminus E_M} \phi f^* \omega_2^1$ above would be given by

$$\sum_{e \subset \partial M} \int_{\mathring{e}} \phi(d\bar{\mu}_e - k ds) - \sum_{p \in \partial M} (\llbracket \mu \rrbracket|_p + \pi - \sum_{T \ni p} \theta_p^T)\phi(p),$$

where $k = \langle \nabla_\tau \vec{n}, \tau \rangle = -\mathrm{I\!I}_{e}{}^{2}_{1}(\tau)$ is the geodesic curvature of e and $ds = \tau^\flat$ is the induced Riemannian length form on e.

Note also that, although the numbers $\bar{\mu}_e$ and m_p are not well defined, $\llbracket \mu \rrbracket|_p$ can often be known in practical scenarios because we may impose constraints on the smooth metric that g is meant to approximate. Sometimes these constraints are dictated by topology. For instance, if we know that M is a manifold with boundary but no corners, then $\llbracket \mu \rrbracket|_p = 0$ for all p, because $\bar{\mu}$ must not have any discontinuities in the smooth metric. The form $d\bar{\mu}_e$, meanwhile, is actually well-defined for any compatible frame on any manifold.

The distributional Gauss-Bonnet functional can be used to derive the Gauss-Bonnet theorem for compact 2-dimensional Regge manifolds:

Theorem 5. *The following equation is true:*

$$\sum_{T \subseteq M} \int_{\mathring{T}} K^T dA + \sum_{\mathring{e} \subset \mathring{M}} \int_{\mathring{e}} \llbracket k \rrbracket ds + \sum_{p \in \mathring{M}} \Theta_p + \sum_{e \subset \partial M} \int_{\mathring{e}} k ds + \sum_{p \in \partial M} (\pi - \sum_{T \ni p} \theta_p^T) = 2\pi\chi(M).$$

Note that, in the case that g is a smooth metric, this theorem reduces to the classical Gauss-Bonnet theorem. Also note that we are using different sign conventions for the geodesic curvature k than some authors do.

Proof. We will prove this theorem by calculating $\langle\langle K_{\text{dist}}, 1 \rangle\rangle$. Note that, since $[\omega, \omega] = 0$ in the case that $n = 2$, we have (for any compatible frame f obtained from a smooth frame $f(0)$ as in Theorem 3)

$$\begin{aligned}
\langle\langle K_{\text{dist}}, 1\rangle\rangle &= \langle\langle f^*\Omega_{\text{dist}}, \frac{1}{2}w_1^2\rangle\rangle + \sum_{e\subset\partial M}\int_{\mathring{e}} d\bar{\mu}_e - \sum_{p\in\partial M}[\![\mu]\!]|_p \\
&= \langle\langle df^*\omega, \frac{1}{2}w_1^2\rangle\rangle + \sum_{e\subset\partial M}\int_{\mathring{e}} d\bar{\mu}_e - \sum_{p\in\partial M}[\![\mu]\!]|_p \\
&= \sum_{T\subseteq M}\int_{\mathring{T}}\langle f^{T^*}\omega \wedge \frac{1}{2}w_1^2 \otimes d(1)\rangle + \sum_{e\subset\partial M}\int_{\mathring{e}} d\bar{\mu}_e - \sum_{p\in\partial M}[\![\mu]\!]|_p \\
&= \sum_{e\subset\partial M}\int_{\mathring{e}} d\bar{\mu}_e - \sum_{p\in\partial M}[\![\mu]\!]|_p.
\end{aligned}$$

Meanwhile, we also have

$$\begin{aligned}
\langle\langle K_{\text{dist}}, 1\rangle\rangle = &\sum_{T\subseteq M}\int_{\mathring{T}} K\,dA + \sum_{\mathring{e}\subseteq\mathring{M}}\int_{\mathring{e}}[\![k ds]\!] + \sum_{p\in\mathring{M}}\Theta_p \\
&+ \sum_{e\subset\partial M}\int_{\mathring{e}} k ds + \sum_{p\in\partial M}\left(\pi - \sum_{T\ni p}\theta_p^T\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{T\subseteq M}\int_{\mathring{T}} K^T dA + \sum_{\mathring{e}\subseteq\mathring{M}}\int_{\mathring{e}}[\![k ds]\!] + \sum_{p\in\mathring{M}}\Theta_p + \sum_{e\subset\partial M}\int_{\mathring{e}} k ds + \sum_{p\in\partial M}\left(\pi - \sum_{T\ni p}\theta_p^T\right) \\
&= \sum_{e\subset\partial M}\int_{\mathring{e}} d\bar{\mu}_e - \sum_{p\in\partial M}[\![\mu]\!]|_p.
\end{aligned}$$

The theorem is almost proved. All we need now is to show that $\sum_{e\subset\partial M}\int_{\mathring{e}} d\bar{\mu}_e - \sum_{p\in\partial M}[\![\mu]\!]|_p = 2\pi\chi(M)$. Note that $\bar{\mu}$ and m_p are the same quantities for the smooth $g(0)$-orthonormal frame $f(0)$. The Gauss-Bonnet theorem for smooth metrics implies that, if we were to evaluate the distributional Gauss-Bonnet functional for the smooth metric $g(0)$ and smooth frame $f(0)$, we must get $2\pi\chi(M)$ for this term. □

Chapter 3

Finite Elements on Regge Manifolds

3.1 Introduction

The analysis of finite element methods on manifolds is a fascinating and quite difficult problem. It is key to producing efficient algorithms for solving PDEs which naturally lie on a surface or which have a natural metric associated to them, such as small vibrations of a curved surface [30], diffusion on a curved surface [23], and thin-film or shallow-water approximations of the Navier-Stokes equations on curved spaces (such as the surface of the Earth) [53]. Numerical discretizations of such PDEs that automatically respect key topological, analytic, and geometric aspects of their domain, such as those appearing in the Finite Element Exterior Calculus (FEEC) framework [2, 3], are highly desirable due to their wide applicability. Although robust convergence theorems have been developed for various types of discretizations of PDEs on Riemannian manifolds and approximations of Riemannian manifolds [21, 22, 36], typically these theorems have been directed toward embedded hypersurfaces in $\mathbb{R}^n$, where the metric is inherited from Euclidean space. (A notable exception is [17], where compact manifolds are treated without reference to an embedding.) However, this restricts the class of manifolds which may be treated, in particular it excludes $(n-1)$-dimensional manifolds which do not possess a natural embedding (or any embedding at all) into $\mathbb{R}^n$.

Recently, Martin Licht has described commuting cochain projections for FEEC on manifolds [43], building upon the foundational work of Holst and Stern [36] and providing the first (to our knowledge) complete proof of the convergence of a finite element method for the Hodge-Laplace problem on manifolds. Our work mostly runs in parallel, with the notable exception that we take the existence of commuting cochain projections as given. However, our treatment of mesh quality measures as well as the setup of the computational problem diverge. We treat the true metric and the approximate metric as existing on the same smooth manifold M and perform all the analysis intrinsically. The approximate metric is a *Regge metric*, that is, a piecewise-smooth metric with some continuity properties. Much of this paper is devoted to building and using machinery for finite elements on manifolds with Regge metrics.

Our initial motivation for studying intrinsic approximations of the Hodge-Laplace problem stemmed from trying to solve another problem that, at the outset, seemed much simpler than it was. On a 2-dimensional Riemannian manifold M which has trivial homology, there is a unique frame such that the corresponding connection form is co-exact. This means it is possible to pose the problem of finding the corresponding connection form as a Hodge-Laplace problem on M with an appropriate source term and boundary conditions. Most interestingly the form, but not its corresponding frame, can be computed only knowing the Gaussian curvature of M and the angle that the frame makes with the normal vector on ∂M. We show how to do this in Section 4. Then we prove convergence of a corresponding numerical scheme in Section 5.

Regge metrics, especially their associated quantities such as curvature and connections, have been studied extensively [6, 14, 15, 27–29, 31–33], often in the context of numerical general relativity and nonlinear elasticity [16, 42, 49, 52]. We are unaware of any existing work using a Regge metric to approximate the Hodge-Laplace equation with a metric-dependent source term as we have done (although Regge calculus has been used for discrete approximations of the Laplacian in the physics literature [12]), however it is a relatively simple matter to specialize the abstract theory of geometric variational crimes once the necessary background is established. The geometric error incurred depends only on quantities intrinsic to M, and local coordinates may be used to take advantage of existing finite element spaces and their approximation properties. We take this up in Section 3.

One key thing that distinguishes our analysis from prior work is our adherence to intrinsic calculations. We take the point of view that all convergence results for a system posed in a manifold should be stated in terms of quantities intrinsic to the manifold, wherever possible avoiding the use of coordinate-dependent expressions. This means that much of the basic theory, as well as scaling arguments, is much more involved than it otherwise would be. It is our belief, however, that these are necessary steps to decouple as much as possible the description of the computational mesh from the geometry of the problem. In Section 3.2 we state and/or prove some elementary properties of the L^2, $H(d)$ (meaning those L^2 differential forms which have an L^2 exterior derivative), and H^1 spaces of differential forms on manifolds with Regge metric and the trace and inverse inequalities for curved elements, as well as fairly ad-hoc, though in our context quite natural, definitions of shape regularity and quasi-uniformity of simplicial meshes. Our bounds do not depend on any embedding of the manifold into an ambient space or any coordinate-dependent expressions, although we do also provide weaker bounds in terms of coordinate-dependent expressions.

Computing connection forms. As we alluded to above, finding/justifying the appropriate discretization of the connection form problem was a deceptively involved process. The difficulty stems from the source term and boundary conditions, which must be treated with great care to produce a well-posed and convergent numerical discretization. Our efforts spurred us to write down a generalized Gauss-Bonnet Theorem for manifolds with Regge metrics in Chapter 2 of this dissertation. With the appropriate source term and boundary conditions, the convergence analysis of our numerical scheme can be carried out with the help of existing work on the distributional scalar curvature of manifolds with Regge metrics. In Section 3.5 we rigorously prove the convergence of the computed connection form in our intrinsic setting. We briefly demonstrate numerical convergence for a simple model problem at the end.

It is natural to wonder whether the computed connection form has any practical uses. In fact it does; our preliminary work indicates that one can leverage it to compute the spectrum of the Bochner Laplacian. (Recall that the Bochner Laplacian, which differs from the Hodge Laplacian, is constructed by composing the covariant derivative with its L^2-adjoint.) We plan to pursue this in future work.

Notation. In this chapter, we are concerned primarily with simplicial complexes, meaning every polytope in the mesh $\mathcal{T}$ (see Section 1.3) is a simplex.

Frequently we will take norms of matrices of 1-forms/matrix-valued 1-forms. The notation $\|\cdot\|_{\_,g}$ will mean the norm of type $\_$ (e.g., the matrix 2-norm $\|\cdot\|_2$, Frobenius norm $\|\cdot\|_F$, ∞-norm $\|\cdot\|_\infty$, or maximum-entry norm $\|\cdot\|_m$) maximized over the set of all norm-1 input vectors V, as measured by g. Specifically, $\|A|_x\|_{\_,g} = \sup_{V \in T_x(M), V \neq 0} \frac{\|A(V)\|_\_}{\|V\|_g}$.

A quantity in double brackets $\llbracket X \rrbracket$ has double meaning: when the quantity is evaluated along a facet shared by two simplices, $\llbracket X \rrbracket := X^+ - X^-$, where X^+ is evaluated using the metric on one side and X^- is evaluated using the metric on the other side, both with the corresponding induced orientations on the boundaries of each simplex. In the case that the quantity is evaluated along a positively oriented edge that lies in the boundary of only one simplex, $\llbracket X \rrbracket := X^+$. In the case that the quantity is evaluated at a corner p of a 2-dimensional triangle T, $\llbracket X|_T \rrbracket := X^+ - X^-$ where X^+ is the quantity evaluated on the "counterclockwise" side and X^- is the quantity evaluated on the "clockwise" side intersecting at p, with the order given by the orientation on T. In one instance, the jump notation is used to refer to the jump of a scalar quantity across a corner on the boundary in the same way.

For two normed spaces X and Y, $\|\cdot\|_{\mathcal{L}(X,Y)}$ is the operator norm on linear maps $X \to Y$.

3.2 Sobolev Spaces on Manifolds with Regge Metrics

3.2.1 Regge Metrics and Their L^2, $H(d)$, and Broken H^1 Spaces

In this section we collect some properties of spaces of differential forms on manifolds with Regge metrics. The development is quite similar to those in [43, 56], but adapted for Regge metrics. We will therefore omit some details for brevity.

We begin by describing the inner product induced by the metric g_T on $\Lambda^k_x(T)$ for every $T \subseteq M, x \in T, k \in \mathbb{N}$. This inner product in turn is induced from the inner product on $T^*_x(T)$:

$$\langle \alpha, \beta \rangle := \alpha(\beta^\sharp),$$

where $\beta^\sharp$ is the unique element of $T_x(T)$ such that, for all $V \in T_x(T)$, $g_T(\beta^\sharp, V) = \beta(V)$. The inner product on $\Lambda^k_x(T)$ is then defined by

$$\langle \alpha^1 \wedge \cdots \wedge \alpha^k, \beta^1 \wedge \cdots \wedge \beta^k \rangle := \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) \prod_{i=1}^{k} \langle \alpha^i, \beta^{\sigma(i)} \rangle$$

and extending multilinearly. This inner product induces the usual corresponding norm $\|\alpha\| := \sqrt{\langle \alpha, \alpha \rangle}$. Notably, the inner product of covectors is dual to the contraction of vector fields: $\alpha(X_1, \ldots, X_k) = \langle \alpha, X_1^\flat \wedge \cdots \wedge X_k^\flat \rangle$, where $\flat$ denotes the inverse of $\sharp$, so the induced norm is the same as the operator norm on $(\bigwedge^k T_x(T))^* \cong \Lambda^k_x(T)$.

Another, often more convenient, way to write the inner product of two alternating k-covectors is to use the Hodge star, which is defined in terms of the aforementioned inner product. We set $\star\alpha$ to be the $(n-k)$-covector such that $\langle \beta, \alpha \rangle dV = \beta \wedge \star\alpha$ for all $\beta \in \Lambda^k_x(T)$, where dV is the volume form induced by g. This map is clearly an invertible isometry, and its inverse is in fact $(-1)^{k(n-k)}\star$.

Lastly we note that the natural tensor product metric $\langle \alpha \otimes \beta, \gamma \otimes \delta \rangle := \langle \alpha, \gamma \rangle \langle \beta, \delta \rangle$ can be used to extend this inner product to $T_x^*(T)^{\otimes p} \otimes \Lambda_x^k(T)$. We will use the shorthand $\Lambda_x^{p,k}(T) := T_x^*(T)^{\otimes p} \otimes \Lambda_x^k(T)$. Similarly, $\Lambda^{p,k}(T) := T^*(T)^{\otimes p} \otimes \Lambda^k(T)$. The space of smooth sections $T \to \Lambda^{p,k}(T)$ will be denoted $C^\infty \Omega^{p,k}(T)$, and the subspace of compactly supported smooth sections $T \to \Lambda^{p,k}(T)$ will be denoted $C_c^\infty \Omega^{p,k}(T)$.

A set $X \subseteq M$ is Σ-*measurable* if $\phi(X \cap U)$ is Lebesgue measurable for each coordinate chart (U, ϕ) of M. The condition of being measurable is preserved by smooth coordinate changes, so this is a well-defined σ-algebra Σ on M, and it is additionally a Borel σ-algebra. A Σ-*measurable* k-*form* is a section $M \to \Lambda^k(M)$ whose coefficients in each coordinate chart are measurable functions $\mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$. Again this condition is invariant under smooth coordinate changes, so it makes sense.

A Σ-measurable k-form α is *locally integrable* if $\sum_{T \subseteq M} \int_T \|\alpha \wedge \phi\| dV < \infty$ for all smooth compactly supported test forms $\phi \in C_c^\infty \Omega^{n-k}(M)$, or equivalently, if each coefficient of $\alpha|_T$ is locally integrable for any simplex $T \subset M$, and for any coordinates on T. The set of locally integrable k-forms will be called $L^1_{\text{loc}} \Omega^k(M)$, and they enjoy the property that $\int_M \alpha \wedge \phi$ is well-defined for any of the test forms. Note that which metric is used to test local integrability does not matter, as $\|\alpha \wedge \phi\|_{g_1} dV_{g_1} = \|\alpha \wedge \phi\|_{g_2} dV_{g_2}$ a.e. for any g_1, g_2. Likewise the integral $\int_T \alpha \wedge \phi$ on each n-dimensional simplex $T \subseteq M$ is defined intrinsically and is independent of any particular metric, because $\alpha \wedge \phi$ is a multiple of the smooth volume form by a Lebesgue integrable function when pulled back along the embedding map $\hat{\Delta}^n \tilde{\to} T \subseteq M$.

Remark 3. *When one considers open, bounded domains in* $\mathbb{R}^n$, *compact support implies vanishing on the boundary. However in our case,* M *can be compact, or can be noncompact but include its boundary. Compact support of a smooth form in this case is used to guarantee a well-defined, finite integral but does not imply anything about boundary conditions. We will focus on compact manifolds with corners rather than open domains in* $\mathbb{R}^n$, *because open domains cannot support a finite triangulation, and the Hodge theory for noncompact,* *incomplete manifolds is substantially more complicated [59]. Focusing in this way means that some definitions below appear slightly nonstandard; for instance, the set of compactly supported smooth functions is dense in the space $H(d)\Omega^0(M,g)$ defined below.*

If α, β are locally integrable k-forms, then in particular they are locally integrable when restricted to each simplex $T \subseteq M$. Because the Regge metric g restricts to the smooth metric g_T on T, the usual theory of Sobolev spaces of functions carries over almost unchanged to the space of tensor-valued differential forms on T [56, p. 30]. The space $L^2\Omega^{p,k}(T,g)$ is the completion of $C_c^\infty\Omega^{p,k}(T)$ under the norm induced by the inner product

$$\langle \alpha, \beta \rangle_{L^2(T,g)} := \int_{\mathring{T}} \langle \alpha, \beta \rangle dV.$$

Of particular interest at this level is the operator $\nabla : C^\infty\Omega^{p,k}(T) \to C^\infty\Omega^{p+1,k}(T)$. For a form $\alpha \in C^\infty\Omega^{p,k}(T)$ and a vector field Y on T, the covariant derivative $\nabla_Y \alpha$ is defined by

$$(\nabla_Y \alpha)(X_1, \ldots, X_{p+k}) = Y(\alpha(X_1, \ldots, X_{p+k})) - \sum_j \alpha(X_1, \ldots, \nabla_Y X_j, \ldots, X_{p+k}).$$

It is easily verified that $\nabla_Y \alpha$ is a $T^*(T)^{\otimes p}$-valued differential k-form and $\nabla_{fY}\alpha = f\nabla_Y \alpha$. Therefore $\nabla\alpha$ is a smooth section of $T^*(T)^{\otimes p+1} \otimes \Lambda^k(T)$.

This operator is by definition bounded on the space $H^1\Omega^{p,k}(T,g)$, which is the completion of $C^\infty\Omega^{p,k}(T)$ with the norm

$$\|\alpha\|_{H^1(T,g)} := \sqrt{\|\alpha\|^2_{L^2(T,g)} + \|\nabla\alpha\|_{L^2(T,g)}}.$$

With the $L^2(T,g)$ spaces and inner products defined for individual top-dimensional simplices T, it is possible to define the $L^2(M,g)$ inner product as simply the sum over all top-dimensional simplices in the triangulation, i.e., $\langle \alpha, \beta \rangle_{L^2(M,g)} := \sum_{T \subseteq M} \langle \alpha, \beta \rangle_{L^2(T,g)}$. The space $L^2\Omega^{p,k}(M,g)$ is defined as the completion of $C_c^\infty\Omega^{p,k}(M)$ under the induced norm.

It is tempting to define the H^1 inner product on M in a similar way, and such a product is useful, but we avoid calling it "the H^1 inner product" on M, because due to discontinuities of the metric g, a form α such that $\nabla\alpha|_T \in L^2\Omega^{1,k}(T,g)$ for each $T \subseteq M$ does not necessarily have a weak covariant derivative that is in $L^2\Omega^{1,k}(M,g)$. Instead it should properly be referred to as a "broken H^1 inner product", although we only use it in the context of a single simplex. Properly defining weak covariant derivatives is outside of the scope of this chapter.

The exterior differential operator d and its formal dual $\delta = (-1)^{n(k-1)-1} \star d \star$ can be defined weakly for locally integrable forms, but only on certain test forms, namely those who have no tangential (respectively, normal) component on the boundary. Formally, we define

$$C_c^\infty\Omega_{\mathbf{t}}^k(M) := \{\phi \in C_c^\infty\Omega^k(M) : i_{\partial M}^*\phi = 0\}.$$

Smooth tangential-free forms are dense in $L^2\Omega^k(M,g)$. To show this, we just need to show that any form $\alpha \in C_c^\infty\Omega^k(M)$ can be approximated by forms in $C_c^\infty\Omega_{\mathbf{t}}^k(M)$. If ρ_ϵ is a smooth compactly supported bump function which is equal to 1 on $\partial M \cap \operatorname{supp}(\alpha)$, is less than or equal to 1 on $\mathring{M}$, and such that the support of ρ_ϵ has volume ϵ, then $(1-\rho_\epsilon)\alpha \in C_c^\infty\Omega_{\mathbf{t}}^{p,k}(M)$ and $\|(1-\rho_\epsilon)\alpha - \alpha\|_{L^2(M,g)} = \|\rho_\epsilon\alpha\|_{L^2(M,g)} \leq \epsilon^{1/2}\sup_{x\in M}\|\alpha|_x\|_g$, which converges to zero as $\epsilon \to 0$. Such bump functions can be constructed because $\partial M \cap \operatorname{supp}(\alpha)$ is compact.

If $\alpha \in L^1_{\text{loc}}\Omega^k(M)$, then the distributional exterior derivative of α will be defined by

$$\forall\phi \in C_c^\infty\Omega_{\mathbf{t}}^{n-k-1}(M), \;\; \langle\langle d\alpha, \phi\rangle\rangle := (-1)^{k+1}\int_M \alpha \wedge d\phi.$$

When α is a smooth form, $\langle\langle d\alpha, \phi\rangle\rangle = \int_M d\alpha \wedge \phi$. We define the $H(d)$ norm for forms $\alpha \in L^2\Omega^k(M,g)$ by

$$\|\alpha\|_{H(d)(M,g)} := \|\alpha\|_{L^2(M,g)} + \sup_{\phi \in C_c^\infty\Omega_{\mathbf{t}}^{n-k-1}(M)\setminus\{0\}} \frac{\langle\langle d\alpha, \phi\rangle\rangle}{\|\phi\|_{L^2(M,g)}}.$$

The space $H(d)\Omega^k(M,g) \subset L^2\Omega^k(M,g)$ is defined as the closure of $C_c^\infty\Omega^k(M)$ with the topology induced by the $H(d)$ norm. Note that if $\alpha \in H(d)\Omega^k(M,g)$, then $d\alpha \in L^2\Omega^{k+1}(M,g)$, because $d\alpha$ as a functional is bounded in the L^2 norm on the dense subset $C_c^\infty\Omega_{\mathbf{t}}^{n-k-1}(M)$.

The map $d : H(d)\Omega^k(M, g) \to L^2\Omega^{k+1}(M, g)$ is continuous by definition and agrees with its smooth counterpart when evaluated on smooth forms. $H(d)\Omega^k(M, g)$ therefore has a natural Hilbert space structure $\langle \alpha, \beta\rangle_{H(d)(M,g)} := \langle \alpha, \beta\rangle_{L^2(M,g)} + \langle d\alpha, d\beta\rangle_{L^2(M,g)}$. We will see later that $H(d)\Omega^k(M, g)$, as a topological space, depends on the metric only up to quasi-isometry, meaning the "small-scale" differences in geometry are inconsequential. (See Section 3.2.2 for the definition of quasi-isometry.)

With the distributional exterior derivative d and the $H(d)$ space defined, the distributional codifferential δ and $H(\delta)$ can be defined fairly easily, with $\delta := (-1)^{n(k-1)-1} \star d\star$ and $H(\delta)\Omega^k(M, g) := \star^{-1}(H(d)\Omega^{n-k}(M, g))$. The following formula is just integration by parts for smooth forms, and the left-hand side remains well-defined if $\alpha \in H(d)\Omega^k(M, g)$ and $\beta \in H(d)\Omega^{n-k-1}(M, g)$:

$$\int_M d\alpha \wedge \beta + (-1)^k \int_M \alpha \wedge d\beta = \int_{\partial M} i^*_{\partial M}(\alpha \wedge \beta). \tag{3.1}$$

So, while $i^*_{\partial M}\alpha$ is not necessarily well-defined, there can still be a notion of "trace zero" if $k < n$. When (3.1) is zero for all β we will say that $\alpha \in H(d)\Omega^k_{\mathbf{t}}(M, g)$. We will also set $H(\delta)\Omega^k_{\mathbf{n}}(M, g) := \star^{-1}(H(d)\Omega^{n-k}_{\mathbf{t}}(M, g))$. As one would expect, as long as $\alpha \in H(d)\Omega^k(M, g)$ and $\beta \in H(\delta)\Omega^{k+1}(M, g)$, we have

$$\langle d\alpha, \beta\rangle_{L^2(M,g)} = \langle \alpha, \delta\beta\rangle_{L^2(M,g)}$$

if $\alpha \in H(d)\Omega^k_{\mathbf{t}}(M, g)$ or $\beta \in H(\delta)\Omega^{k+1}_{\mathbf{n}}(M, g)$. In other words, $H(\delta)\Omega^k_{\mathbf{n}}(M, g)$ is the L^2 dual of $H(d)\Omega^k(M, g)$ and $H(d)\Omega^k_{\mathbf{t}}(M, g)$ is the dual of $H(\delta)\Omega^k(M, g)$.

The spaces $H(d)\Omega^k(M, g)$ and $H(\delta)\Omega^k(M, g)$ contain subspaces consisting of compactly supported piecewise-smooth forms. It is useful to understand what these subspaces look like.

If $i^*_e[\![\alpha]\!] \neq 0$ for some $(n-1)$-simplex e that is not contained in ∂M, that is if the tangential components of α are not continuous, then test forms ϕ can be constructed so that $\|\phi\|_{L^2(M,g)}$ is arbitrarily small but $\langle\langle d\alpha, \phi\rangle\rangle$ is arbitrarily large, which would mean that $\|\langle\langle d\alpha, \cdot\rangle\rangle\|_{L^2\Omega^{n-k-1}(M,g)^*} = \infty$. Therefore the compactly supported piecewise-smooth elements of $H(d)\Omega^k(M,g)$ are those which are continuous in their tangential components across n-simplex boundaries. This gives some intuition as to why the space $H(d)\Omega^k(M,g)$, and the map d, are independent of the small-scale geometry of g, since this is a purely topological condition.

Similarly, the compactly supported piecewise-smooth elements of $H(\delta)\Omega^k(M,g)$ are those which are continuous in their normal components, as defined by g. These will generally be discontinuous and will depend on the small-scale geometry of g, meaning they are significantly less convenient to work with when designing a finite element method.

3.2.2 Bounds on Sobolev Norms from Differing Metrics

A measurement of the "difference" between two Regge metrics is an important ingredient for analyzing approximation properties of metrics and PDE problems depending on them. The notion of *quasi-isometry*, fundamental in geometric group theory [34] and long studied in relation to approximations (both analytical and metrical) of Riemannian manifolds [38], provide a natural and powerful tool to this end.

Given two Regge metrics $(\mathcal{T}_1, g_1)$ and $(\mathcal{T}_2, g_2)$ on the same manifold M, and a smooth submanifold $\Omega \subseteq M$, we define

$$C_{g_1,g_2}(\Omega) := \operatorname*{ess\,sup}_{x\in\Omega,\ v\in T_x\Omega\setminus\{0\}} \frac{\|v\|_{g_1}}{\|v\|_{g_2}} \tag{3.2}$$

and

$$D_{g_1,g_2}(\Omega) := \operatorname*{ess\,sup}_{x\in\Omega} \|dV_{g_1}|_x\|_{g_2}, \tag{3.3}$$

where dV_{g_1} is understood to mean the g_1-volume form of Ω, and a set X has measure 0 if $\mu_{\text{Lebesgue}}(\phi(X \cap U)) = 0$ for any coordinate chart (U, ϕ) of Ω. We define $C_{g_1,g_2}(\emptyset)$ and $D_{g_1,g_2}(\emptyset)$ to be zero for convenience, as these quantities are always nonnegative.

When $\Omega = \bigcup_i \Omega_i$ is a finite union of smooth submanifolds (such as the boundary of a simplex), $C_{g_1,g_2}(\Omega) := \sup_i C_{g_1,g_2}(\Omega_i)$ and likewise for $D_{g_1,g_2}(\Omega)$.

If $0 < C_{g_1,g_2}(M) < \infty$ and $0 < C_{g_2,g_1}(M) < \infty$, we will say that $(M, \mathcal{T}_1, g_1)$ and $(M, \mathcal{T}_2, g_2)$ are *quasi-isometric*. This induces an equivalence relation on the set of smooth manifolds with Regge metrics. Quasi-isometry can be interpreted to mean that two manifolds are "asymptotically the same". It is very straightforward to show that if $\mathcal{T}_1$ and $\mathcal{T}_2$ are both finite triangulations, then g_1 and g_2 are quasi-isometric, and hence a compact manifold M supports only one quasi-isometry class.

The following small lemma is very useful for showing results related to $C_{g_1,g_2}(\Omega)$.

Lemma 5. *Let V be a vector space and $(V, \|\cdot\|_{g_1})$, $(V, \|\cdot\|_{g_2})$ be two reflexive Banach space structures on V. By abuse of notation, also use the same symbols for the induced operator norms on V^*. Then:*

$$\sup_{\alpha\in V^*,\, \alpha\neq 0} \frac{\|\alpha\|_{g_1}}{\|\alpha\|_{g_2}} = \sup_{v\in V,\, v\neq 0} \frac{\|v\|_{g_2}}{\|v\|_{g_1}} \tag{3.4}$$

and

$$\inf_{v\in V,\, v\neq 0} \frac{\|v\|_{g_2}}{\|v\|_{g_1}} = \inf_{\alpha\in V^*,\, \alpha\neq 0} \frac{\|\alpha\|_{g_1}}{\|\alpha\|_{g_2}}. \tag{3.5}$$

Proof. Let $\alpha \in V^*$. Then:

$$\begin{aligned} \|\alpha\|_{g_1} &= \sup_{v\neq 0} \frac{|\alpha(v)|}{\|v\|_{g_1}} = \sup_{v\neq 0} \frac{\left|\alpha\left(\frac{v}{\|v\|_{g_2}}\right)\right|}{\frac{\|v\|_{g_1}}{\|v\|_{g_2}}} \\ &\leq \left(\sup_{v\neq 0} \frac{\|v\|_{g_2}}{\|v\|_{g_1}}\right)\left(\sup_{v\neq 0} \frac{|\alpha(v)|}{\|v\|_{g_2}}\right) = \sup_{v\in V,\, v\neq 0} \frac{\|v\|_{g_2}}{\|v\|_{g_1}} \|\alpha\|_{g_2}. \end{aligned}$$

Since the operator norms on V^{**} are isometric to the original norms on V, the same reasoning also shows that if $v \in V = V^{**}$ then $\|v\|_{g_2} \leq \sup_{\alpha\in V^*,\, \alpha\neq 0} \frac{\|\alpha\|_{g_1}}{\|\alpha\|_{g_2}} \|v\|_{g_1}$, proving equality in the first statement.

The second statement follows from the fact that $\inf_x \frac{a(x)}{b(x)} = \frac{1}{\sup_x \frac{b(x)}{a(x)}}$. ☐

As one might expect, L^2 spaces of forms depend only on the quasi-isometry class of the metric.

Theorem 6. *If $(\mathcal{T}_1, g_1)$ and $(\mathcal{T}_2, g_2)$ are two Regge metrics on the same manifold M, then*

1. $\frac{1}{C_{g_1,g_2}(M)^n} \leq D_{g_2,g_1}(M) \leq C_{g_2,g_1}(M)^n$,
2. $\forall \alpha \in L^2\Omega^{p,k}(M, g_2),\ \|\alpha\|^2_{L^2(M,g_1)} \leq \binom{n}{k} n^p C_{g_2,g_1}(M)^{2p+2k} D_{g_1,g_2}(M) \|\alpha\|^2_{L^2(M,g_2)}$.

Proof. We will establish a uniform bound on $\|\alpha\|_{g_1}$ when $\alpha \in T^*_x(M)^p \otimes \Lambda^k_x(M),\ \|\alpha\|_{g_2} = 1$, with a special case for $k = n,\ p = 0$ (in which case $\alpha = dV_{g_2}$). In this proof, $x \in M$ is always a point around which g_1 and g_2 are both smooth in a neighborhood. We will also use $\{F_i\}_{i=1}^n$ and $\{\theta^i\}_{i=1}^n$ to refer to a frame and coframe that are orthonormal in the metric g_1 and dual to each other, and $\{\tilde{F}_i\}_{i=1}^n$ and $\{\tilde{\theta}^i\}_{i=1}^n$ to refer to a frame and coframe that are orthonormal in the metric g_2 and dual to each other. The two frames will also induce the same orientation on M.

Since the covector norm is the same as the operator norm, $\|dV_{g_2}\|_{g_1} = \|\tilde{\theta}^1 \wedge \cdots \wedge \tilde{\theta}^n\|_{g_1} = \det[\tilde{\theta}^i(F_j)]$. Let $\tilde{\Theta}$ be the matrix defined by $\tilde{\Theta}_{ij} = \tilde{\theta}^i(F_j)$. If $u, w \in \mathbb{R}^n$, then $u \cdot \tilde{\Theta} w = \langle u^i \tilde{F}_i, w^j F_j \rangle_{g_2}$. The eigenvalues of the matrix $\tilde{\Theta}$ have magnitude bounded above by $\|\tilde{\Theta}\|_2 = \sup_{w \in \mathbb{R}^n \setminus \{0\}} \frac{\|\tilde{\Theta} w\|_2}{\|w\|_2} = \sup_{u,w \in \mathbb{R}^n \setminus \{0\}} \frac{u \cdot \tilde{\Theta} w}{\|u\|_2 \|w\|_2}$. However, writing $u' = u^i \tilde{F}_i$ and $w' = w^j F_j$, this expression is the same as $\sup_{u',w' \in T_x(M) \setminus \{0\}} \frac{\langle u', w' \rangle_{g_2}}{\|u'\|_{g_2} \|w'\|_{g_1}} = \sup_{w' \in T_x(M) \setminus \{0\}} \frac{\|w'\|_{g_2}}{\|w'\|_{g_1}}$. Likewise, the eigenvalues of $\tilde{\Theta}$ have magnitude bounded below by $\inf_{w \in \mathbb{R}^n \setminus \{0\}} \frac{\|\tilde{\Theta} w\|_2}{\|w\|_2} = \inf_{w' \in T_x(M) \setminus \{0\}} \frac{\|w'\|_{g_2}}{\|w'\|_{g_1}}$. The determinant is the product of eigenvalues, so we have

$$\begin{aligned}
\frac{1}{C_{g_1,g_2}(M)^n} &\leq \left(\frac{1}{\sup_{v \in T_x(M),\ v \neq 0} \frac{\|v\|_{g_1}}{\|v\|_{g_2}}} \right)^n \\
&= \left(\inf_{v \in T_x(M),\ v \neq 0} \frac{\|v\|_{g_2}}{\|v\|_{g_1}} \right)^n \leq \|dV_{g_2}\|_{g_1} \leq \left(\sup_{v \in T_x(M),\ v \neq 0} \frac{\|v\|_{g_2}}{\|v\|_{g_1}} \right)^n = C_{g_2,g_1}(M)^n,
\end{aligned}$$

proving part 1 of the theorem.

To bound $\|\alpha\|_{g_1}$ for general p and k, we express α in coordinates: $\alpha = \sum_{I,J} \alpha_{IJ} (\tilde{\theta}^{j_1} \otimes \cdots \otimes \tilde{\theta}^{j_p}) \otimes \tilde{\theta}^{i_1} \wedge \cdots \wedge \tilde{\theta}^{i_k}$, where I ranges over all increasing multi-indices $(i_1, \ldots, i_k) \in [1, n]^k$ and J ranges over all tuples of indices $(j_1, \ldots, j_p) \in [1, n]^p$.

Since $\|\alpha\|_{g_2} = 1$, we have $\sum_{I,J} \alpha_{IJ}^2 = 1$. For convenience, in the following lines we will use $\tilde{\theta}^I$ to refer to $\tilde{\theta}^{i_1} \wedge \cdots \wedge \tilde{\theta}^{i_k}$ and $\otimes^J \tilde{\theta}$ to refer to $\tilde{\theta}^{j_1} \otimes \cdots \otimes \tilde{\theta}^{j_p}$. We have

$$\begin{aligned}
\langle \alpha, \alpha \rangle_{g_1} &= \sum_{I,J,L,M} \alpha_{IJ}\alpha_{LM} \langle \otimes^J \tilde{\theta}, \otimes^M \tilde{\theta} \rangle_{g_1} \langle \tilde{\theta}^I, \tilde{\theta}^L \rangle_{g_1} \\
&= \sum_{I,J,L,M} \alpha_{IJ}\alpha_{LM} \prod_{s=1}^{p} \langle \tilde{\theta}^{j_s}, \tilde{\theta}^{m_s} \rangle_{g_1} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) \prod_{t=1}^{k} \langle \tilde{\theta}^{i_t}, \tilde{\theta}^{l_{\sigma(t)}} \rangle_{g_1}.
\end{aligned}$$

The inner product $\langle \tilde{\theta}^I, \tilde{\theta}^L \rangle_{g_1}$ above is equal to the determinant of the $k \times k$ matrix $\tilde{\Theta}^{IL}_{tr} = \langle \tilde{\theta}^{i_t}, \tilde{\theta}^{l_r} \rangle_{g_1}$. Since $\tilde{\theta}^i = \tilde{\theta}^i(F_a)\theta^a$, we have $\tilde{\Theta}^{IL} = E^I (E^L)^\intercal$, where $E^I_{ta} = \tilde{\theta}^{i_t}(F_a)$ and likewise for E^L. As before, for $u \in \mathbb{R}^k$ and $w \in \mathbb{R}^n$, we have $u \cdot E^I w = \langle u^t \tilde{F}_{i_t}, w^a F_a \rangle_{g_2}$, and $\|E^I\|_2 = \sup_{u' \in \operatorname{span}(\tilde{F}_{i_1}, \ldots, \tilde{F}_{i_k}) \setminus \{0\}, w' \in T_x(M) \setminus \{0\}} \frac{\langle u', w' \rangle_{g_2}}{\|u'\|_{g_2} \|w'\|_{g_1}}$. By expanding the range of u', we get $\|E^I\|_2 \leq \sup_{w' \in T_x(M) \setminus \{0\}} \frac{\|w'\|_{g_2}}{\|w'\|_{g_1}}$. Similarly, $w \cdot (E^L)^\intercal u = \langle w^b F_b, u^r \tilde{F}_{l_r} \rangle_{g_1}$, so $\|(E^L)^\intercal\|_2 \leq \sup_{u' \in T_x(M) \setminus \{0\}} \frac{\|u'\|_{g_2}}{\|u'\|_{g_1}}$. Therefore

$$|\langle \tilde{\theta}^I, \tilde{\theta}^L \rangle_{g_1}| = |\det(\tilde{\Theta}^{IL})| \leq \|\tilde{\Theta}^{IL}\|_2^k \leq \|E^I\|_2^k \|(E^L)^\intercal\|_2^k \leq \sup_{v \in T_x(M) \setminus \{0\}} \left(\frac{\|v\|_{g_2}}{\|v\|_{g_1}} \right)^{2k}.$$

The terms $\langle \tilde{\theta}^{j_s}, \tilde{\theta}^{m_s} \rangle_{g_1}$, meanwhile, are simply bounded above by $\sup_{\tilde{\theta} \in T_x^*(M) \setminus \{0\}} \frac{\|\tilde{\theta}\|_{g_1}^2}{\|\tilde{\theta}\|_{g_2}^2}$ by the Cauchy-Schwarz inequality.

Therefore

$$\begin{aligned}
|\langle \alpha, \alpha \rangle_{g_1}| &\leq \sup_{\tilde{\theta} \in T_x^*(M) \setminus \{0\}} \left(\frac{\|\tilde{\theta}\|_{g_1}}{\|\tilde{\theta}\|_{g_2}} \right)^{2p} \sup_{v \in T_x(M) \setminus \{0\}} \left(\frac{\|v\|_{g_2}}{\|v\|_{g_1}} \right)^{2k} \sum_{I,J,L,M} |\alpha_{IJ}\alpha_{LM}| \\
&= \sup_{v \in T_x(M) \setminus \{0\}} \left(\frac{\|v\|_{g_2}}{\|v\|_{g_1}} \right)^{2k+2p} \sum_{I,J} |\alpha_{IJ}| \sum_{L,M} |\alpha_{LM}| \\
&\leq C_{g_2,g_1}(M)^{2(k+p)} \binom{n}{k} n^p,
\end{aligned}$$

since $\sqrt{\binom{n}{k} n^p}$ is the maximum 1-norm of all unit vectors in $\mathbb{R}^{\binom{n}{k} n^p}$. Thus, for all $\alpha \in T_x^*(M)^p \otimes \Lambda_x^k(M)$, $\|\alpha\|_{g_1}^2 \leq \binom{n}{k} n^p C_{g_2,g_1}(M)^{2(k+p)} \|\alpha\|_{g_2}^2$.

Part 2 of the theorem is now almost immediate. Let $\alpha \in C_c^\infty\Omega^{p,k}(M)$. Then

$$\begin{aligned}\|\alpha\|^2_{L^2(M,g_1)} &= \sum_{T\in\mathcal{T}_1}\int_T \|\alpha\|^2_{g_1} dV_{g_1}\\ &\leq \binom{n}{k} n^p \sum_{T\in\mathcal{T}_1,\, T'\in\mathcal{T}_2} C_{g_2,g_1}(T\cap T')^{2k+2p}\int_{T\cap T'}\|\alpha\|^2_{g_2}\|dV_{g_1}\|_{g_2}dV_{g_2}\\ &\leq \binom{n}{k} n^p \sum_{T'\in\mathcal{T}_2} C_{g_2,g_1}(T')^{2k+2p} D_{g_1,g_2}(T')\int_{T'}\|\alpha\|^2_{g_2}dV_{g_2}\\ &\leq \binom{n}{k} n^p C_{g_2,g_1}(M)^{2k+2p}D_{g_1,g_2}(M)\|\alpha\|^2_{L^2(M,g_2)}.\end{aligned}$$

Since $C_c^\infty\Omega^{p,k}(M)$ is dense in $L^2\Omega^{p,k}(M,g_2)$, the same inequality holds for all $\alpha \in L^2\Omega^{p,k}(M,g_2)$. □

This theorem will be useful for proving the scaling properties for the trace and inverse inequalities. It could be tightened substantially by using more information about the metrics, but $C_{g_1,g_2}(M)$ and $D_{g_1,g_2}(M)$ are easy to compute and widely applicable. It also has an immediate corollary:

Corollary 2. *If g_1 and g_2 are two quasi-isometric Regge metrics on M, then, for all p,k, $L^2\Omega^{p,k}(M,g_1) = L^2\Omega^{p,k}(M,g_2)$ and $H(d)\Omega^k(M,g_1) = H(d)\Omega^k(M,g_2)$ as topological vector spaces.*

In what follows we will frequently need to make use of comparisons between the covariant derivatives of forms. The following lemma establishes a relationship between covariant derivatives in different metrics, which is key to proving the trace and inverse inequalities.

Lemma 6. *Given two Regge metrics g and g' on a manifold M and their Levi-Civita connections ∇ and ∇', $\nabla - \nabla'$ is a bundle morphism $T^*(T\cap T')^p \otimes \Lambda^k(T\cap T') \to T^*(T\cap T')^{p+1}\otimes\Lambda^k(T\cap T')$ on each $T, T' \subseteq M$ where g and g' are smooth respectively. Furthermore, if $U \subseteq T\cap T'$ is measurable and contained in a coordinate chart with coordinates $\{x^i\}_{i=1}^n$,* *then*

$$\begin{aligned}\|\nabla - \nabla'\|_{\mathcal{L}(L^2\Omega^{p,k}(U,g),L^2\Omega^{p+1,k}(U,g))} \leq \frac{3(k+p)n^{\frac{7}{2}}}{2}&\|G^{-1}\|_{L^\infty(U)}\|G\|_{L^\infty(U)}\\ &\cdot\big(\|G^{-1}-G'^{-1}\|_{L^\infty(U)}\|dG\|_{L^\infty(U,g)}\\ &\quad+\|G'^{-1}\|_{L^\infty(U)}\|dG-dG'\|_{L^\infty(U,g)}\big),\end{aligned}$$

where $G_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle_g$ *is the coordinate expression for* g *and* G' *is defined similarly. For a matrix-valued one-form* A, $\|A\|_{L^\infty(U,g)} := \operatorname{ess\,sup}_{x\in U} \|A(x)\|_{2,g}$ *and for a matrix-valued function,* $\|A\|_{L^\infty(U)} := \operatorname{ess\,sup}_{x\in U} \|A(x)\|_2$.

Remark 4. *While the fact that the difference between Christoffel symbols is bounded by* $\|G - G'\|$ *and* $\|dG - dG'\|$ *is fairly obvious, this theorem makes explicit how they relate to bounds on the* L^2 *operator norm of* $\nabla - \nabla'$. *Additionally, if* g *is piecewise flat, there exists a coordinate chart* $U \subseteq T \cap T'$ *around any point* $x \in T \cap T'$ *such that* $G = \lambda I$ *for a constant* $\lambda > 0$ *and* $\|G'^{-1}\|_{L^\infty(U)} = 1$, *yielding a much simpler, and intrinsic, expression:*

$$\|\nabla - \nabla'\|_{\mathcal{L}(L^2\Omega^{p,k}(U,g),L^2\Omega^{p+1,k}(U,g))} \leq \frac{3(k+p)n^{\frac{7}{2}}}{2}\|dG'\|_{L^\infty(U,g)} = \frac{3(k+p)n^{\frac{7}{2}}}{2}\|\nabla g'\|_{L^\infty(U,g)} \tag{3.6}$$

Here g' *is understood to be a piecewise smooth* $(0,2)$*-tensor and* $\nabla g'$ *a* $(0,3)$*-tensor, and* $\|\nabla g'\|_{L^\infty(U,g)} := \operatorname{ess\,sup}_{x\in U} \|\nabla g'|_x\|_g$.

Proof of Lemma 6. To see that $\nabla - \nabla'$ is a bundle morphism for all $x \in T \cap T'$, all that is necessary is to calculate

$$(\nabla - \nabla')_V(fW) = df(V)W + f\nabla_V W - df(V)W - f\nabla'_V W = f(\nabla - \nabla')_V W.$$

What this shows is that $\nabla - \nabla'$ is $C^\infty(T)$-linear, which by definition means $(\nabla - \nabla')|_x : T_x(T\cap T') \otimes T_x(T\cap T') \to T_x(T\cap T')$ is a well-defined linear map for all $x \in T \cap T'$. Clearly this means $(\nabla - \nabla')|_x : T^*_x(T \cap T)^p \otimes \Lambda^k_x(T \cap T') \to T^*_x(T \cap T')^{p+1} \otimes \Lambda^k_x(T \cap T')$ is also a well-defined linear map.

Next we will find a bound on $|\Gamma^i_{jk} - \Gamma'^i_{jk}|$, where Γ^i_{jk} and Γ'^i_{jk} are the Christoffel symbols associated with g and g', respectively. First let E be the Cholesky factor of G^{-1}, i.e. $E^\intercal E = G^{-1}$, so that $E_i = \sum_j E_{ij}\frac{\partial}{\partial x^j}$ is a g-orthonormal frame. Then

$$\begin{aligned}
&|\Gamma^i_{jk} - \Gamma'^i_{jk}| = \frac{1}{2}|G^{il}(G_{lj,k} + G_{lk,j} - G_{jk,l}) - G'^{il}(G'_{lj,k} + G'_{lk,j} - G'_{jk,l})| \\
&= \frac{1}{2}|(G^{il} - G'^{il})(G_{lj,k} + G_{lk,j} - G_{jk,l}) + G'^{il}(G_{lj,k} + G_{lk,j} - G_{jk,l} - G'_{lj,k} - G'_{lk,j} + G'_{jk,l})| \\
&\leq \frac{3}{2}\Bigg[\left(\max_i \sum_j |G^{-1}_{ij} - G'^{-1}_{ij}|\right) \max_{i,j,k}\left|dG_{ij}\left(\frac{\partial}{\partial x^k}\right)\right| \\
&\qquad + \left(\max_i \sum_j |G'^{-1}_{ij}|\right) \max_{i,j,k}\left|(dG - dG')_{ij}\left(\frac{\partial}{\partial x^k}\right)\right|\Bigg] \\
&\leq \frac{3}{2}\Bigg(\|G^{-1} - G'^{-1}\|_\infty \max_{i,j,k}\left|dG_{ij}\left(\sum_l E^{-1}_{kl}E_l\right)\right| \\
&\qquad + \|G'^{-1}\|_\infty \max_{i,j,k}\left|(dG - dG')_{ij}\left(\sum_l E^{-1}_{kl}E_l\right)\right|\Bigg) \\
&\leq \frac{3}{2}\|E^{-1}\|_\infty(\|G^{-1} - G'^{-1}\|_\infty\|dG\|_{m,g} + \|G'^{-1}\|_\infty\|dG - dG'\|_{m,g}).
\end{aligned}$$

This allows us to bound $\|(\nabla - \nabla')_{E_i}E_j\|_g$:

$$\begin{aligned}
\|(\nabla - \nabla')_{E_i}E_j|_x\|_g &= \left\|\sum_{k,l} E_{ik}E_{jl}(\nabla - \nabla')_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^l}\right\|_g \\
&= \left\|\sum_k E_{ik}\sum_l E_{jl}(\Gamma^o_{kl} - \Gamma'^o_{kl})\frac{\partial}{\partial x^o}\right\|_g \\
&\leq \|E\|^2_\infty \max_{k,l,o}|\Gamma^o_{kl} - \Gamma'^o_{kl}|\left\|\sum_p E^{-1}_{op}E_p\right\|_g \\
&\leq \frac{3}{2}\|E\|^2_\infty\|E^{-1}\|^2_\infty(\|G^{-1} - G'^{-1}\|_\infty\|dG\|_{m,g} + \|G'^{-1}\|_\infty\|dG - dG'\|_{m,g}) \\
&\leq \frac{3n^{\frac{5}{2}}}{2}\|E\|^2_2\|E^{-1}\|^2_2(\|G^{-1} - G'^{-1}\|_2\|dG\|_{2,g} + \|G'^{-1}\|_2\|dG - dG'\|_{2,g}) \\
&= \frac{3n^{\frac{5}{2}}}{2}\|G^{-1}\|_2\|G\|_2(\|G^{-1} - G'^{-1}\|_2\|dG\|_{2,g} + \|G'^{-1}\|_2\|dG - dG'\|_{2,g}).
\end{aligned}$$

In the second to last line, we used the fact that $\|\cdot\|_m \leq \|\cdot\|_2$ and $\|\cdot\|_\infty \leq \sqrt{n}\|\cdot\|_2$ for matrices. In the last line, we used the fact that $\|E\|^2_2 = \|G^{-1}\|_2$ since E is the Cholesky factor of G^{-1}.

Then we can calculate

$$\begin{aligned}\|(\nabla - \nabla')|_x\|_{2,g} &= \sup_{V,W \in T_x(T\cap T')\setminus\{0\}} \frac{\|(\nabla-\nabla')_V W\|_g}{\|V\|_g\|W\|_g} \leq n \max_{i,j} \|(\nabla-\nabla')_{E_i}E_j\|_g \\ &\leq \frac{3n^{\frac{7}{2}}}{2}\|G^{-1}\|_2\|G\|_2(\|G^{-1}-G'^{-1}\|_2\|dG\|_{2,g} + \|G'^{-1}\|_2\|dG-dG'\|_{2,g}).\end{aligned}$$

Lastly, we will calculate the pointwise norm of $(\nabla-\nabla')|_x\alpha$ for an arbitrary $\alpha \in T^*_x(T\cap T')^p \otimes \Lambda^k_x(T\cap T')$:

$$\begin{aligned}&\|(\nabla - \nabla')|_x\alpha\|_g = \sup_{V,W_1,\ldots,W_{k+p}\in T_x(T\cap T')\setminus\{0\}} \frac{|\sum_j \alpha(W_1,\ldots,(\nabla-\nabla')_V W_j,\ldots,W_{k+p})|}{\|V\|_g\|W_1\|_g\ldots\|W_{k+p}\|_g} \\ &\leq \sum_j \sup_{V,W_1,\ldots,W_{k+p}} \frac{\|(\nabla-\nabla')_V W_j\|_g}{\|V\|_g\|W_j\|_g} \frac{\left|\alpha(W_1,\ldots,W_{j-1},\frac{(\nabla-\nabla')_V W_j}{\|(\nabla-\nabla')_V W_j\|_g},W_{j+1},\ldots,W_{k+p})\right|}{\|W_1\|_g\ldots\widehat{\|W_j\|_g}\ldots\|W_{k+p}\|_g} \\ &\leq \|(\nabla-\nabla')|_x\|_{2,g}\sum_j \sup_{W_1,\ldots,W_{k+p}} \frac{|\alpha(W_1,\ldots,W_{k+p})|}{\|W_1\|_g\ldots\|W_{k+p}\|_g} \\ &\leq (k+p)\|(\nabla-\nabla')|_x\|_{2,g}\|\alpha\|_g.\end{aligned}$$

Therefore the $L^2(U,g)$ operator norm of $\nabla - \nabla' : C^\infty\Omega^{p,k}(U) \to C^\infty\Omega^{p+1,k}(U)$ is less than or equal to $(k+p)\operatorname{ess\,sup}_{x\in U}\|(\nabla-\nabla')|_x\|_{2,g}$, giving the required bound.

□

3.2.3 Trace and Inverse Inequalities

With all preparations done, we are finally ready to prove trace and inverse inequalities for simplices endowed with an arbitrary Riemannian metric. The usual scaling arguments are more complicated than usual, so complete proofs are provided. In this section, T is a single n-simplex of $\mathcal{T}$, so g is smooth on T.

Theorem 7 (H^1 Trace Inequality for Riemannian Simplices)**.**
Suppose $\alpha \in H^1\Omega^{p,k}(T,g)$, and assume that there exists an orientation-preserving diffeomorphism $f : \hat{T} \to T$ where $\hat{T} \subset \mathbb{R}^n$ is the standard simplex. Then $i^*_{\partial T}\alpha$ is well-defined

and belongs to $L^2\Omega^{p,k}(\partial T, g)$*, and there exists* $\hat{C}$ *depending only on* n*,* p*, and* k *such that*

$$\begin{aligned}
&\|i^*_{\partial T}\alpha\|_{L^2(\partial T,g)} \\
&\quad\le \hat{C} C_{g,\hat{\delta}}(T)^{k+p} C_{\hat{\delta},g}(T)^{k+p+\frac{1}{2}} \sqrt{D_{g,\hat{\delta}}(T) D_{\hat{\delta},g}(T)} \\
&\qquad \cdot \Bigg[\left(1 + C_{\hat{\delta},g}(T)^{k+p+1} C_{g,\hat{\delta}}(T)^{k+p+1} \sqrt{D_{\hat{\delta},g}(T) D_{g,\hat{\delta}}(T)} \|\hat{\nabla} g\|_{L^\infty(T,\hat{\delta})}\right) \|\alpha\|_{L^2(T,g)} \\
&\qquad\quad + C_{g,\hat{\delta}}(T) \|\nabla\alpha\|_{L^2(T,g)}\Bigg],
\end{aligned}$$

where $\hat{\delta} = f^{-*}\delta$ *is the metric induced by* f *and the standard Euclidean metric* δ *on* $\hat{T}$*, and* $\hat{\nabla}$ *is the associated connection.*

Remark 5. *In the case that* $p = k = 0$*,* $\hat{\nabla}\alpha = d\alpha = \nabla\alpha$*, and the term involving* $\|\hat{\nabla} g\|$ *can be removed as it is measuring the difference between* ∇ *and* $\hat{\nabla}$*.*

Proof of theorem 7. We will take as given that there is a trace theorem for differential forms on $(\hat{T}, \delta)$. This is essentially because a form in this manifold is in H^1 if and only if each of its coefficients is in H^1 in the standard coordinates, and a trace theorem holds for each coefficient independently. Thus, setting $\hat{\alpha} = f^*\alpha$, $i^*_{\partial\hat{T}}\hat{\alpha}$ is well-defined and

$$\|i^*_{\partial\hat{T}}\hat{\alpha}\|^2_{L^2(\partial\hat{T},\delta)} \le \hat{C}\|\hat{\alpha}\|^2_{H^1(\hat{T},\delta)}. \tag{3.7}$$

Let $i^*_{\partial T}\alpha := f^{-*} i^*_{\partial\hat{T}}\hat{\alpha}$. This definition is consistent when α is continuous, because f restricts to an embedding $e \hookrightarrow M$ for each facet $e \subset \partial T$. We will bound both sides of this inequality. Firstly, by Theorem 6,

$$\begin{aligned}
\|i^*_{\partial\hat{T}}\hat{\alpha}\|^2_{L^2(\partial\hat{T},\delta)} &= \int_{\partial\hat{T}} \|f^* i^*_{\partial T}\alpha\|^2_\delta dS_\delta = \int_{\partial T} \|i^*_{\partial T}\alpha\|^2_{f^{-*}\delta} f^{-*} dS_\delta \\
&= \int_{\partial T} \|i^*_{\partial T}\alpha\|^2_{\hat{\delta}} dS_{\hat{\delta}} = \|i^*_{\partial T}\alpha\|^2_{L^2(\partial T,\hat{\delta})} \\
&\ge \frac{1}{N_1^2 C_{\hat{\delta},g}(\partial T)^{2k+2p} D_{g,\hat{\delta}}(\partial T)} \|i^*_{\partial T}\alpha\|^2_{L^2(\partial T,g)},
\end{aligned}$$

where N_1 is a constant depending only on n, p, and k. Secondly,

$$\begin{aligned}
\|\hat{\alpha}\|^2_{H^1(\hat{T},\delta)} &= \int_{\hat{T}} \|f^*\alpha\|^2_{\delta} dV_{\delta} + \int_{\hat{T}} \|\nabla_{\delta} f^*\alpha\|^2_{\delta} dV_{\delta} \\
&= \int_T \|\alpha\|^2_{\hat{\delta}} dV_{\hat{\delta}} + \int_T \|\hat{\nabla}\alpha\|^2_{\hat{\delta}} dV_{\hat{\delta}} \\
&= \|\alpha\|^2_{L^2(T,\hat{\delta})} + \|\hat{\nabla}\alpha\|^2_{L^2(T,\hat{\delta})} \\
&\leq N_2^2 C_{g,\hat{\delta}}(T)^{2k+2p} D_{\hat{\delta},g}(T) \left(\|\alpha\|^2_{L^2(T,g)} + C_{g,\hat{\delta}}(T)^2 \|\hat{\nabla}\alpha\|^2_{L^2(T,g)} \right),
\end{aligned}$$

where N_2 is another constant depending only on n, p, and k. Plugging both of these inequalities into inequality (3.7), we obtain

$$\begin{aligned}
\|i^*_{\partial T}\alpha\|^2_{L^2(\partial T,g)} \leq \hat{C} N_1^2 N_2^2 C_{\hat{\delta},g}(\partial T)^{2(k+p)} C_{g,\hat{\delta}}(T)^{2(k+p)} D_{\hat{\delta},g}(T) D_{g,\hat{\delta}}(\partial T) \\
\cdot \left(\|\alpha\|^2_{L^2(T,g)} + C_{g,\hat{\delta}}(T)^2 \|\hat{\nabla}\alpha\|^2_{L^2(T,g)} \right).
\end{aligned} \tag{3.8}$$

It is clearly the case that $C_{g_1,g_2}(\partial T) \leq C_{g_1,g_2}(T)$ for any two continuous metrics g_1, g_2 on T. We can also bound $D_{g,\hat{\delta}}(\partial T)$ in terms of $D_{g,\hat{\delta}}(T)$. In a relatively open neighborhood $U \subset T$ of a point x_0 of ∂T which is contained in a smooth component of ∂T, dV_g can be written as $\theta^1 \wedge \star_g \theta^1$, where $\|\theta^1\|_g = 1$ and $\ker \theta^1|_x = T_x \partial T$ for all $x \in U \cap \partial T$. Evidently, $i^*_{\partial T} \star_g \theta^1$ is the g volume form on $U \cap \partial T$.

At each x in U, we have the following inequality:

$$\begin{aligned}
&\sup_{W_1,\ldots,W_n \in T_xT \setminus \{0\}} \frac{|dV_g(W_1,\ldots,W_n)|}{\|W_1\|_{\hat{\delta}} \cdots \|W_n\|_{\hat{\delta}}} \\
&\qquad \geq \sup_{W_2,\ldots,W_n \in \ker \theta^1|_x \setminus \{0\}} \sup_{W_1 \notin \ker \theta^1|_x} \frac{|\theta^1(W_1)| \cdot |\star_g \theta^1(W_2,\ldots,W_n)|}{\|W_1\|_{\hat{\delta}} \|W_2\|_{\hat{\delta}} \cdots \|W_n\|_{\hat{\delta}}} \\
&\qquad = \|\theta^1|_x\|_{\hat{\delta}} \| \star_g \theta^1|_x\|_{\hat{\delta}}.
\end{aligned}$$

Therefore, we have

$$D_{g,\hat{\delta}}(U) \geq D_{g,\hat{\delta}}(U \cap \partial T) \inf_{x \in U} \|\theta^1|_x\|_{\hat{\delta}} \geq \frac{D_{g,\hat{\delta}}(U \cap \partial T)}{C_{\hat{\delta},g}(U)}.$$

This local inequality implies a global inequality

$$D_{g,\hat{\delta}}(\partial T) \leq C_{\hat{\delta},g}(T) D_{g,\hat{\delta}}(T).$$

Plugging this into inequality (3.8), we derive

$$\begin{aligned}\|i^*_{\partial T}\alpha\|^2_{L^2(\partial T,g)} \leq \hat{C}^2 N_1^2 N_2^2 C_{\hat{\delta},g}(T)^{2(k+p)+1} C_{g,\hat{\delta}}(T)^{2(k+p)} \\ \cdot D_{\hat{\delta},g}(T) D_{g,\hat{\delta}}(T) \left(\|\alpha\|^2_{L^2(T,g)} + C_{g,\hat{\delta}}(T)^2 \|\hat{\nabla}\alpha\|^2_{L^2(T,g)}\right),\end{aligned}$$

or, more conveniently,

$$\begin{aligned}\|i^*_{\partial T}\alpha\|_{L^2(\partial T,g)} \leq \hat{C} N_1 N_2 C_{\hat{\delta},g}(T)^{k+p+\frac{1}{2}} C_{g,\hat{\delta}}(T)^{k+p} \\ \cdot \sqrt{D_{\hat{\delta},g}(T) D_{g,\hat{\delta}}(T)} \left(\|\alpha\|_{L^2(T,g)} + C_{g,\hat{\delta}}(T) \|\hat{\nabla}\alpha\|_{L^2(T,g)}\right). \qquad (3.9)\end{aligned}$$

Lastly, we can use Lemma 6 to assert that

$$\|\hat{\nabla}\alpha\|_{L^2(T,g)} \leq \|\nabla\alpha\|_{L^2(T,g)} + \|\hat{\nabla} - \nabla\|_{\mathcal{L}(L^2\Omega^{p,k}(T,g), L^2\Omega^{p+1,k}(T,g))} \|\alpha\|_{L^2(T,g)}.$$

Since $\hat{\delta}$ is flat on T, we have by inequality 3.6 that $\|\hat{\nabla} - \nabla\|_{\mathcal{L}(L^2\Omega^{p,k}(T,\hat{\delta}), L^2\Omega^{p+1,k}(T,\hat{\delta})} \leq \frac{3(k+p)n^{\frac{7}{2}}}{2} \|\hat{\nabla} g\|_{L^\infty(T,\hat{\delta})}$. To apply this inequality, we'll use

$$\begin{aligned}&\|\hat{\nabla} - \nabla\|_{\mathcal{L}(L^2\Omega^{p,k}(T,g), L^2\Omega^{p+1,k}(T,g))} \\ &\leq \frac{\sup_{\beta \in L^2\Omega^{p+1,k}(T,g)\setminus\{0\}} \frac{\|\beta\|_{L^2(T,g)}}{\|\beta\|_{L^2(T,\hat{\delta})}}}{\inf_{\beta \in L^2\Omega^{p,k}(T,g)\setminus\{0\}} \frac{\|\beta\|_{L^2(T,g)}}{\|\beta\|_{L^2(T,\hat{\delta})}}} \|\hat{\nabla} - \nabla\|_{\mathcal{L}(L^2\Omega^{p,k}(T,\hat{\delta}), L^2\Omega^{p+1,k}(T,\hat{\delta}))} \\ &\leq \hat{C}_1 \hat{C}_2 C_{\hat{\delta},g}(T)^{k+p+1} C^{k+p}_{g,\hat{\delta}}(T) \sqrt{D_{\hat{\delta},g}(T) D_{g,\hat{\delta}}(T)} \|\hat{\nabla} - \nabla\|_{\mathcal{L}(L^2\Omega^{p,k}(T,\hat{\delta}), L^2\Omega^{p+1,k}(T,\hat{\delta}))} \\ &\leq \hat{C}_1 \hat{C}_2 \frac{3(k+p)n^{\frac{7}{2}}}{2} C_{\hat{\delta},g}(T)^{k+p+1} C_{g,\hat{\delta}}(T)^{k+p} \sqrt{D_{\hat{\delta},g}(T) D_{g,\hat{\delta}}(T)} \|\hat{\nabla} g\|_{L^\infty(T,\hat{\delta})},\end{aligned}$$

where $\hat{C}_1$ and $\hat{C}_2$ depend only on n, p, and k.

Plugging this into inequality 3.9 and grouping together constant multiples, we obtain the desired bound.

□

Next is the well-known inverse inequality.

Theorem 8 (Inverse Inequality for Riemannian Simplices)**.** *Let $\hat{T}$, T, δ, $\hat{\delta}$, g, ∇, $\hat{\nabla}$, and f be as in Theorem 7, and let $\hat{V} \subset H(d)\Omega^k(\hat{T},\delta)$ be a finite-dimensional subspace and let*

$V_h = f^{-*}(\hat{V}) \subset H(d)\Omega^k(T,g)$. Then there exists $\hat{C}'$ depending only on n, k, and $\hat{V}$ such that for all $u_h \in V_h$,

$$\|du_h\|_{L^2(T,g)} \leq \hat{C}' C_{\hat{\delta},g}(T)^{k+1} C_{g,\hat{\delta}}(T)^k \sqrt{D_{\hat{\delta},g}(T) D_{g,\hat{\delta}}(T)} \|u_h\|_{L^2(T,g)}.$$

Likewise, if $\hat{V} \subset H^1\Omega^{p,k}(\hat{T},\delta)$ is finite-dimensional and $V_h = f^{-*}(\hat{V}) \subset H^1\Omega^{p,k}(T,g)$, then there exists $\hat{C}$ depending only on n, p, k, and $\hat{V}$ such that for all $u_h \in V_h$,

$$\|\nabla u_h\|_{L^2(T,g)} \leq \hat{C} C_{\hat{\delta},g}(T)^{k+p+1} C_{g,\hat{\delta}}(T)^{k+p} \sqrt{D_{\hat{\delta},g}(T) D_{g,\hat{\delta}}(T)} [1 + \|\hat{\nabla} g\|_{L^2(T,\hat{\delta})}] \|u_h\|_{L^2(T,g)}.$$

Proof. The proofs of both claims are quite similar, so we will prove the second claim and then explain how the same proof idea carries over to the first claim. Let $\hat{u}_h = f^* u_h$. We start with the observation that any two norms on the finite-dimensional subspace $\hat{V}$ are equivalent, so there exists $\hat{C}$ independent of $\hat{u}_h$ such that

$$\|\nabla_\delta \hat{u}_h\|_{L^2(\hat{T},\delta)} \leq \|\hat{u}_h\|_{H^1(\hat{T},\delta)} \leq \hat{C} \|\hat{u}_h\|_{L^2(\hat{T},\delta)}. \tag{3.10}$$

We'll bound both sides.

$$\begin{aligned}
\|\nabla_\delta \hat{u}_h\|_{L^2(\hat{T},\delta)} &= \|\hat{\nabla} u_h\|_{L^2(T,\hat{\delta})} \\
&\geq \frac{1}{N_1 C_{\hat{\delta},g}(T)^{k+p+1} \sqrt{D_{g,\hat{\delta}}(T)}} \|\hat{\nabla} u_h\|_{L^2(T,g)} \\
&\geq \frac{1}{N_1 C_{\hat{\delta},g}(T)^{k+p+1} \sqrt{D_{g,\hat{\delta}}(T)}} \\
&\qquad \cdot \left[\|\nabla u_h\|_{L^2(T,g)} - \|\hat{\nabla} - \nabla\|_{\mathcal{L}(L^2\Omega^{p,k}(T,g), L^2\Omega^{p+1,k}(T,g))} \|u_h\|_{L^2(T,g)} \right], \\
\|\hat{u}_h\|_{L^2(\hat{T},\delta)} &= \|u_h\|_{L^2(T,\hat{\delta})} \\
&\leq N_2 C_{g,\hat{\delta}}(T)^{k+p} \sqrt{D_{\hat{\delta},g}(T)} \|u_h\|_{L^2(T,g)}.
\end{aligned}$$

Plugging these inequalities into inequality (3.10), we get

$$\begin{aligned}
\|\nabla u_h\|_{L^2(T,g)} \leq \Big[&\hat{C} N_1 N_2 C_{\hat{\delta},g}(T)^{k+p+1} C_{g,\hat{\delta}}(T)^{k+p} \sqrt{D_{\hat{\delta},g}(T) D_{g,\hat{\delta}}(T)} \\
&+ \|\hat{\nabla} - \nabla\|_{\mathcal{L}(L^2\Omega^{p,k}(T,g), L^2\Omega^{p+1,k}(T,g))} \Big] \|u_h\|_{L^2(T,g)}.
\end{aligned}$$

And, just as in the proof of Theorem 7, we can apply the inequality

$$\|\hat{\nabla} - \nabla\|_{\mathcal{L}(L^2\Omega^{p,k}(T,g),L^2\Omega^{p+1,k}(T,g))} \leq \hat{C}_1\hat{C}_2\frac{3(k+p)n^{\frac{7}{2}}}{2}C_{\hat{\delta},g}(T)^{k+p+1}C_{g,\hat{\delta}}(T)^{k+p} \cdot \sqrt{D_{\hat{\delta},g}(T)D_{g,\hat{\delta}}(T)}\|\hat{\nabla}g\|_{L^\infty(T,\hat{\delta})},$$

which gives the desired bound.

The proof of the $H(d)$ inverse inequality is nearly identical with ∇ replaced with d, except that there is no difference between $\hat{d}$ and d. □

It is helpful to consider how these theorems relate to coordinate expressions. If $M = \mathbb{R}^n$ and g is the Euclidean metric then, expressed in coordinates on the reference simplex, we have $\hat{\nabla}g = \hat{\nabla}(JJ^\intercal)$ where $J = df$ is the Jacobian, and $C_{g,\hat{\delta}}(T) = \sup_{x\in\hat{T}}\|J(x)\|_2$, $C_{\hat{\delta},g}(T) = \sup_{x\in\hat{T}}\|J^{-1}(x)\|_2$, $D_{g,\hat{\delta}}(T) = \sup_{x\in\hat{T}}\det(J(x))$, and $D_{\hat{\delta},g}(T) = \sup_{x\in\hat{T}}\det(J^{-1}(x))$. Scaling arguments for the case when g is the Euclidean metric are already well established in the literature, even when f is not affine [7, 39]. However, usually such arguments lack a term analogous to the $\hat{\nabla}g$ term because they consider real-valued functions rather than tensors.

3.2.4 Riemannian Shape-Regularity

The issue of mesh quality on a manifold is subtle, as not only the geometry of a simplex, but also the quality of its embedding map $f_{h,T}$ must be considered. For the purpose of computational ease, we have chosen to introduce new definitions of shape-regularity and shape constant. These are equivalent to the usual definitions if each T is a flat simplex in some Euclidean space and the map $f_{h,T}$ is affine, and they have some similarities in their role in the trace and inverse inequalities.

Definition 4. *A family of manifolds with Regge metrics* $\{(M_h, \mathcal{T}_h, g_h)\}_{h\in S}$ *(where* $S \subset \mathbb{R}$*) has the* Riemannian shape-regularity property *if there exists a number* $0 < K < \infty$ *independent of* h *and orientation-preserving diffeomorphisms* $f_{h,T} : \hat{T} \to T$ *for each* $T \in \mathcal{T}_h$ *such that* $C_{g_h,\hat{\delta}}(T)C_{\hat{\delta},g_h}(T) \leq K$ *for all* $h \in S, T \in \mathcal{T}_h$.

Here, as before, $\hat{T}$ denotes the standard simplex and $\hat{\delta} = f_{h,T}^{-*}\delta$ is the pushforward of the standard metric on $\hat{T}$.

The quantity $C_{g_h,\hat{\delta}}(T)C_{\hat{\delta},g_h}(T)$ simultaneously measures how much g_h differs from a scaled version of the Euclidean metric $\hat{\delta}$, and how much $f_{h,T}$ differs from a pure rigid motion and a scaling. As the next lemma will show, if a family of meshes on a compact manifold is piecewise-affine in some smooth curvilinear coordinates and shape-regular (in the Euclidean sense) when viewed in those coordinates, and the metrics g_h differ from a smooth metric g by a bounded amount, then the family of meshes is also shape-regular in the Riemannian sense.

Lemma 7. *Suppose (M, g) is a compact Riemannian manifold with corners, and let $\{U_i\}_{i=1}^N$ be a finite cover of M by open sets each possessing a smooth embedding $\phi_i : U_i \to \mathbb{R}^{K_i}$ for some integer K_i, and let $V_i \subset U_i$ be another cover where each V_i is closed. If $\{(M, \mathcal{T}_h, g_h)\}_{h\in S}$ is a manifold supporting a family of Regge metrics such that*

1. *there exists $C < \infty$ such that $C_{g,g_h}(M)C_{g_h,g}(M) \leq C$ for all $h \in S$,*
2. *every n-simplex T is contained completely in V_i for some index i,*
3. *for each $h \in S, T \in \mathcal{T}_h$ there exists a one-to-one affine map $\tilde{f}_{h,T} : \mathbb{R}^n \to \mathbb{R}^{K_i}$ such that $\tilde{f}_{h,T}(\hat{T}) = \phi_i(T)$ (by abuse of notation, we'll also use $\tilde{f}_{h,T}$ to refer to the linear map $v \mapsto \tilde{f}_{h,T}(v) - \tilde{f}_{h,T}(0)$),*
4. *there exists $0 < K < \infty$ such that $\dfrac{\sup_{v\neq 0} \frac{\|\tilde{f}_{h,T}v\|}{\|v\|}}{\inf_{w\neq 0} \frac{\|\tilde{f}_{h,T}w\|}{\|w\|}} \leq K$ for all $h \in S, T \in \mathcal{T}_h$,*

then $\{(M, \mathcal{T}_h, g_h)\}_{h\in S}$ has the Riemannian shape-regularity property.

Condition 4 is equivalent to the usual shape-regularity property $\frac{h_T}{\rho_T} \leq K$ in the local Euclidean metric, which involves the Euclidean diameter h_T of T and the Euclidean diameter ρ_T of the largest ball contained in T.

Proof of Lemma 7. Each U_i has an induced flat metric $\delta_{U_i} := \phi_i^*(\sum_j dy^j \otimes dy^j)$, and each simplex $T \subseteq V_i$ has the induced flat metric $\hat{\delta} := (\tilde{f}_{h,T}^{-1} \circ \phi_i)^*(\sum_j dx^j \otimes dx^j)$. For a given $w \in \mathbb{R}^n$ and $x \in T$, we can define a corresponding vector $W = d(\phi_i^{-1} \circ \tilde{f}_{h,T})(w^j \frac{\partial}{\partial x^j}) \in T_x(T)$. Then $\|w\| = \|W\|_{\hat{\delta}}$ and $\|\tilde{f}_{h,T} w\| = \|W\|_{\delta_{U_i}}$. Since any vector $W \in T_x(T)$ can be expressed this way, condition 4 is equivalent to $C_{\delta_{U_i},\hat{\delta}}(T) C_{\hat{\delta},\delta_{U_i}}(T) \leq K$.

Clearly from the definition of $C_{\hat{\delta},g_h}(T)$,

$$C_{\hat{\delta},g_h}(T) \leq C_{g,g_h}(T) C_{\delta_{U_i},g}(T) C_{\hat{\delta},\delta_{U_i}}(T)$$

and

$$C_{g_h,\hat{\delta}}(T) \leq C_{g_h,g}(T) C_{g,\delta_{U_i}}(T) C_{\delta_{U_i},\hat{\delta}}(T).$$

Since V_i is closed and M is compact, V_i is compact, so g and δ_{U_i} are quasi-isometric on V_i. Let $C' := \max_i (C_{g,\delta_{U_i}}(V_i) C_{\delta_{U_i},g}(V_i))$. Then, we have

$$C_{\hat{\delta},g_h}(T) C_{g_h,\hat{\delta}}(T) \leq CC'K,$$

which is exactly what we need for the family to be Riemannian shape-regular, using $f_{h,T} := \phi_i^{-1} \circ \tilde{f}_{h,T}$ for the maps $\hat{T} \to T$. $\square$

We can also introduce a new definition of quasi-uniformity. In order for this definition to be meaningful in the same way it is in the Euclidean case, h can no longer be an abstract parameter. Instead, we will reparameterize the family $\{(M_h, \mathcal{T}_h, g_h)\}_{h \in S}$ so that $\max_{T \subseteq M_h} C_{g_h,\hat{\delta}}(T) = h$. This is analogous to using h to represent the maximum diameter of a simplex.

Definition 5. *Let* $\{(M_h, \mathcal{T}_h, g_h)\}_{h \in S}$ *and the maps* $f_{h,T} : \hat{T} \to M_h$ *be as in Definition (4), such that* $\max_{T \subseteq M_h} C_{g_h,\hat{\delta}}(T) = h$*. If there exists a number* $0 < K' < \infty$ *such that* $C_{\hat{\delta},g_h}(T) \leq K' h^{-1}$ *for all* $h \in S, T \in \mathcal{T}_h$ *then we will say that the family of manifolds and Regge metrics has the* Riemannian Quasi-Uniformity Property*.*

The quantity $C_{\hat{\delta},g_h}(T)$ measures how much the map $f_{h,T}$ can shrink the length of a curve (measured in the g_h metric) compared to its Euclidean length in the standard unit

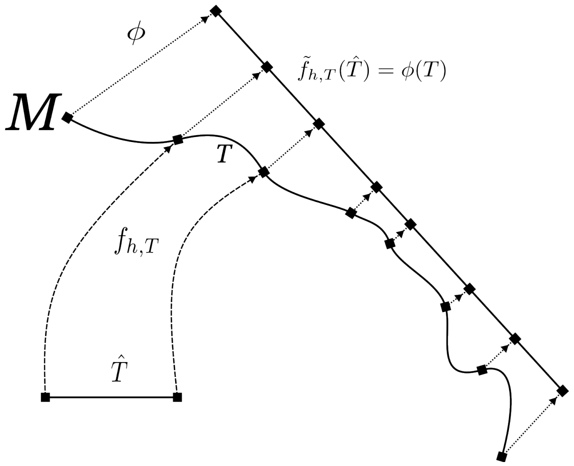

Figure 3.1: Example in which M is a smooth curve in $\mathbb{R}^2$ and $\phi : M \to \mathbb{R}^2$ is a projection onto a straight line. The maps $f_{h,T} : [0,1] \to M$ can be taken as compositions of the affine maps $\tilde{f}_{h,T} : \hat{T} \to \phi(T)$ with the nonlinear map ϕ^{-1}.

simplex. By bounding it from above, we assert that the simplices do not "shrink too fast" relative to h. As before, the usual notion of quasi-uniformity in curvilinear coordinates implies Riemannian-quasi-uniformity.

Lemma 8. *With the assumptions of Lemma 7, and the additional assumptions:*

5. $C_{g,g_h}(T) \le C$ *for all* $h \in S, T \in \mathcal{T}_h$,

6. *there exists* $0 < K' < \infty$ *such that* $0 < \frac{1}{K'}h \le \inf_{v\neq 0} \frac{\|\tilde{f}_{h,T}v\|}{\|v\|}$ *for all* $h \in S, T \in \mathcal{T}_h$,

then $\{(M, \mathcal{T}_h, g_h)\}_{h\in S}$ *has the Riemannian quasi-uniformity property.*

Condition 6 is equivalent to the usual quasi-uniformity property $\frac{1}{K'}h \le \rho_T$.

Proof of lemma 8. We already know that $C_{\hat{\delta},g_h}(T) \le C_{\hat{\delta},\delta_{U_i}}(T) C_{\delta_{U_i},g}(T) C_{g,g_h}(T)$ and $C_{\hat{\delta},\delta_{U_i}}(T) = \sup_{v\neq 0} \frac{\|v\|}{\|\tilde{f}_{h,T}v\|} = \frac{1}{\inf_{v\neq 0} \frac{\|\tilde{f}_{h,T}v\|}{\|v\|}} \le \frac{1}{\frac{1}{K'}h}$. Therefore $C_{\hat{\delta},g_h}(T) \le \max_i C_{\delta_{U_i},g}(V_i) C K' h^{-1}$. $\square$

Remark 6. *The maps ϕ_i do not need to be smooth embeddings on their whole domain, they merely need to be smooth embeddings when restricted to each simplex. An option for generalization would be to instead use a family of maps $\{\phi_{i,h}\}_{h \in S}$ such that $\phi_{i,h} : U_i \to \mathbb{R}^{K_i}$ is a piecewise-smooth embedding for $\mathcal{T}_h$, and there exist smooth embeddings $\phi_i : U_i \to \mathbb{R}^{K_i}$ and a constant C such that $\|d\phi^j_{i,h} - d\phi^j_i\|_{L^\infty(V_i,g)} \leq C$ for all $h \in S$ and each coordinate $j = 1, \ldots, K_i$. If C is small enough, $C_{\delta_{U_i,h},g}(V_i)$ and $C_{g,\delta_{U_i,h}}(V_i)$ are both uniformly bounded, which would suffice to show shape-regularity and quasi-uniformity using the maps $f_{h,T} = \phi^{-1}_{i,h} \circ \hat{f}_{h,T}$. This could be useful, for instance, if M is a smooth hypersurface in $\mathbb{R}^{n+1}$, g is the metric on M inherited from $\mathbb{R}^{n+1}$, and a piecewise-flat triangulated surface is used to approximate the smooth surface. The mesh $\mathcal{T}_h$ could be obtained by projecting from the approximate surface to the true surface, and the maps $\phi_{i,h}$ could be defined as the inverse of this projection map, with codomain $\mathbb{R}^{n+1}$. The nearby smooth maps ϕ_i could each be defined as the inclusion map $M \hookrightarrow \mathbb{R}^{n+1}$. It must be stressed that in this situation, the approximate metrics g_h need not necessarily be obtained from $\phi_{i,h}$, but setting g_h equal to the pullback under $\phi_{i,h}$ of the Euclidean metric on $\mathbb{R}^{n+1}$ would yield a first-order approximation of g.*

To emphasize the roles of the different quantities, we will define $h_T := C_{g_h,\hat{\delta}}(T)$ and $\rho_{h,T} := \frac{1}{C_{\hat{\delta},g_h}(T)}$. The Riemannian shape-regularity property can be rephrased as $\frac{h_T}{\rho_{h,T}} \leq K$ and the Riemannian quasi-uniformity property can be rephrased as ${\rho_{h,T}}^{-1} \leq K'h^{-1}$. We will also distinguish the bound on volume deformation by setting $K_V := \max_{T \subseteq M} D_{g_h,\hat{\delta}}(T) D_{\hat{\delta},g_h}(T)$, since volumes typically transform differently than lengths in the scaling arguments. Note also that $\rho_{h,T} \leq h_T$ since $1 = C_{g_h,g_h}(T) \leq C_{g_h,\hat{\delta}}(T) C_{\hat{\delta},g_h}(T) \leq K$.

The value of Riemannian shape-regularity even in the circumstances when Lemmas (7) and (8) apply is that the resulting estimates for K, K_V, and K' are overestimates unless the coordinate maps happen to be isometric embeddings for the smooth metric g. This discrepancy is especially problematic if g is highly curved.

Corollary 3 (Trace Inequality for Shape-regular Simplicial Complexes)**.** *If* $\{(M_h, \mathcal{T}_h, g_h)\}_{h\in S}$ *is Riemannian shape-regular, then there exists* $\hat{C}$ *independent of* h, T, α *such that, for all* $T \in \mathcal{T}_h$ *and* $\alpha \in H^1\Omega^{p,k}(T, g_h)$,

$$\|i^*_{\partial T}\alpha\|_{L^2(\partial T, g_h)} \leq \hat{C}\sqrt{K_V}K^{k+p}\Big[{\rho_{h,T}}^{-\frac{1}{2}}\Big(1 + K^{k+p+1}\sqrt{K_V}\|\hat{\nabla}g_h\|_{L^\infty(T,\hat{\delta})}\Big)\|\alpha\|_{L^2(T,g_h)} + K^{\frac{1}{2}}h_T^{\frac{1}{2}}\|\nabla_h\alpha\|_{L^2(T,g_h)}\Big].$$

Corollary 4 (Inverse Inequality for Shape-regular Simplicial Complexes)**.** *If* $\{(M_h, \mathcal{T}_h, g_h)\}_{h\in S}$ *is Riemannian shape-regular,* $\hat{V} \subset H(d)\Omega^k(\hat{T}, \delta)$ *is finite-dimensional,* $V_h := f^{-*}_{h,T}(\hat{V})$*, then there exists* $\hat{C}$ *depending only on* $\hat{V}$*,* n*, and* k *such that for all* $u_h \in V_h$,

$$\|du_h\|_{L^2(T,g_h)} \leq \hat{C}\sqrt{K_V}K^{k+1}{\rho_{h,T}}^{-1}\|u_h\|_{L^2(T,g_h)}.$$

Likewise if $\hat{V} \subset H^1\Omega^{p,k}(\hat{T}, \delta)$ *is finite-dimensional,* $V_h := f^{-*}_{h,T}(\hat{V})$*, then there exists* $\hat{C}$ *depending only on* $\hat{V}$*,* n*,* k*, and* p *such that for all* $u_h \in V_h$,

$$\|\nabla_h u_h\|_{L^2(T,g_h)} \leq \hat{C}\sqrt{K_V}K^{k+p}{\rho_{h,T}}^{-1}\left(1 + \|\hat{\nabla}g_h\|_{L^\infty(T,\hat{\delta})}\right)\|u_h\|_{L^2(T,g_h)}.$$

3.3 Error Analysis of the Hodge-Laplace Problem on Manifolds with Regge Metrics

In this section we will work out finite element a priori error analysis for the Hodge-Laplace problem on manifolds with Regge metrics, building on the foundation of geometric variational crimes [36]. This section is mostly a straightforward application of existing theory.

Briefly, we start with a compact smooth Riemannian manifold with corners (M, g) and the L^2-de Rham complex of differential forms, on which the Hodge-Laplace problem can be posed in the strong form (for a source term $f \in L^2\Omega^k(M, g)$):

$$(d\delta + \delta d)u = f \quad \text{on } \mathring{M}, \tag{3.11}$$

$$i^*_{\partial M} \star u = i^*_{\partial M} \star du = 0. \tag{3.12}$$

The space of harmonic forms $\mathfrak{H}^k(M,g) := \{\alpha \in H(d)\Omega^k(M,g) \cap H(\delta)\Omega^k_{\mathbf{n}}(M,g) : d\alpha = \delta\alpha = 0\}$ can be shown to be equal to the closure of the set $\{u \in C^\infty\Omega^k(M) : (d\delta + \delta d)u = 0, i^*_{\partial M} \star u = i^*_{\partial M} \star du = 0\}$ in $H^1\Omega^k(M,g)$ [56]. This is significant as the solution is only unique up to addition by a harmonic form, and the fact that the space of harmonic forms is compactly embedded in L^2 is a key component in proving the well-posedness of weak formulations of the problem.

There is a mixed weak formulation of this problem, which is well-posed [2, 3]: Find $(\sigma, u, p) \in H(d)\Omega^{k-1}(M,g) \times H(d)\Omega^k(M,g) \times \mathfrak{H}^k(M,g)$ such that, for all $(\tau, v, q) \in H(d)\Omega^{k-1}(M,g) \times H(d)\Omega^k(M,g) \times \mathfrak{H}^k(M,g)$,

$$\begin{aligned} \langle \sigma, \tau \rangle_{L^2(M,g)} - \langle d\tau, u \rangle_{L^2(M,g)} &= 0 \\ \langle d\sigma, \tau \rangle_{L^2(M,g)} + \langle du, dv \rangle_{L^2(M,g)} + \langle p, v \rangle_{L^2(M,g)} &= \langle f, v \rangle_{L^2(M,g)} \\ \langle u, q \rangle_{L^2(M,g)} &= 0. \end{aligned} \tag{3.13}$$

Remark 7. *Note that f does not necessarily need to be in $L^2\Omega^k(M,g)$, as the mixed weak formulation is well-posed for any element of $H(d)\Omega^k(M,g)^*$. From now on we will use F to refer to such a functional. While this change is not necessarily meaningful for the strong formulation of the problem, it can still be useful in geometric applications, as we will see in the next section.*

To approximate this boundary value problem, we would like to use a finite-dimensional subcomplex $V_h^k \subset H(d)\Omega^k(M, g_h)$ where g_h is a Regge metric on a triangulation $\mathcal{T}_h$ which approximates g, such that $d(V_h^k) \subset V_h^{k+1}$. We will use V_h^k to refer to this space with the $H(d)\Omega^k(M, g_h)$ norm, and W_h^k to refer to the same space with the $L^2\Omega^k(M, g_h)$ norm. The space of discrete harmonic forms is given by $\mathfrak{H}_h^k := \{u \in V_h^k : du = 0, \langle u, d\eta \rangle_{L^2(M,g_h)} = 0 \forall \eta \in V_h^{k-1}\}$. The discrete weak problem is then: given $F_h \in {V_h^k}^*$, find $(\sigma_h, u_h, p_h) \in$

$V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ such that, for all $(\tau_h, v_h, q_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$,

$$\begin{aligned} \langle \sigma_h, \tau_h \rangle_{L^2(M,g_h)} - \langle d\tau_h, u_h \rangle_{L^2(M,g_h)} &= 0, \\ \langle d\sigma_h, \tau_h \rangle_{L^2(M,g_h)} + \langle du_h, dv_h \rangle_{L^2(M,g_h)} + \langle p_h, v_h \rangle_{L^2(M,g_h)} &= F_h(v_h), \\ \langle u_h, q_h \rangle_{L^2(M,g_h)} &= 0. \end{aligned} \tag{3.14}$$

To analyze the stability and accuracy of this method in the framework of geometric variational crimes, we need to produce bounded cochain maps $i_h^k : W_h^k \to L^2\Omega^k(M,g)$ and $\pi_h^k : H(d)\Omega^k(M,g) \to V_h^k$ such that $i_h^k(W_h^k) \subset H(d)\Omega^k(M,g)$ and $\pi_h^k \circ i_h^k = \mathrm{Id}$. Since the L^2 and $H(d)$ spaces of forms on a compact manifold with corners do not depend (as topological spaces) on the metric, i_h^k can simply be the inclusion map $W_h^k \subset L^2\Omega^k(M,g)$, and π_h^k can be a family of bounded cochain projection operators $H(d)\Omega^k(M,g) \to i_h^k(V_h^k)$. This will induce bounded cochain maps $W_h^k \to L^2\Omega^k(M,g)$ and $H(d)\Omega^k(M,g) \to V_h^k$ with the desired properties so long as $C_{g_h,g}(M)$ and $C_{g,g_h}(M)$ are uniformly bounded with respect to h.

Specifically,

$$\|i_h^k\|_{\mathcal{L}(W_h^k, L^2\Omega^k(M,g))} = \sup_{v_h \in W_h \setminus \{0\}} \frac{\|v_h\|_{L^2\Omega^k(M,g)}}{\|v_h\|_{L^2\Omega^k(M,g_h)}} \leq \sqrt{\binom{n}{k}} C_{g_h,g}(M)^k \sqrt{D_{g,g_h}(M)}$$

and

$$\begin{aligned} \|\pi_h^k\|_{\mathcal{L}(H(d)\Omega^k(M,g), V_h^k)} &= \sup_{v \in H(d)\Omega^k(M,g) \setminus \ker \pi_h^k} \frac{\|\pi_h^k(v)\|_{H(d)(M,g_h)}}{\|\pi_h^k(v)\|_{H(d)(M,g)}} \frac{\|\pi_h^k(v)\|_{H(d)(M,g)}}{\|v\|_{H(d)(M,g)}} \\ &\leq \sup_{v_h \in V_h^k \setminus \{0\}} \frac{\|v_h\|_{H(d)(M,g_h)}}{\|v_h\|_{H(d)(M,g)}} \sup_{v \in H(d)\Omega^k(M,g) \setminus \{0\}} \frac{\|\pi_h^k(v)\|_{H(d)(M,g)}}{\|v\|_{H(d)(M,g)}} \\ \leq \sqrt{\binom{n}{k}} \max(C_{g,g_h}(M)^k, & C_{g,g_h}(M)^{k+1}) \sqrt{D_{g_h,g}(M)} \|\pi_h^k\|_{\mathcal{L}(H(d)\Omega^k(M,g), H(d)\Omega^k(M,g))}. \end{aligned}$$

We will take the existence of a family of commuting projection operators $\pi_h^k : H(d)\Omega^k(M,g) \to i_h^k(V_h^k)$ as given. Such operators have been constructed, mostly for Euclidean domains [18] and somewhat recently for Riemannian manifolds [43]. Their construction is not trivial. However, it is worth noting that the canonical degrees of freedom for the $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$

spaces from finite element exterior calculus [3] are completely independent of the metric when applied to smooth forms, as they only depend on integrals of smooth differential j-forms over j-simplices (for $j \geq k$) and a local homotopy operator κ. The difficulty is in constructing a smoothing operator $H(d)\Omega^k(M,g) \to C^\infty\Omega^k(M)$ that is bounded in the $H(d)$ norm, commutes with the exterior derivative, and preserves boundary conditions.

Theorem 3.9 in [36] establishes that the discrete problem is well-posed with a uniform inf-sup constant, while Corollary 3.11 in [36] establishes that the following error bound holds:

$$\begin{aligned}
&\|\sigma - \sigma_h\|_{H(d)(M,g)} + \|u - u_h\|_{H(d)(M,g)} + \|p - p_h\|_{L^2(M,g)} \\
&\leq C\Bigg[\inf_{\tau \in V_h^{k-1}} \|\sigma - \tau\|_{H(d)(M,g)} + \inf_{v \in V_h^k} \|u - v\|_{H(d)(M,g)} + \inf_{q \in V_h^k} \|p - q\|_{H(d)(M,g)} \\
&\qquad + \inf_{v \in V_h^k} \sup_{r \in \mathfrak{H}^k(M,g)\setminus\{0\}} \frac{\|(Id - \pi_h^k) r\|_{L^2(M,g)}}{\|r\|_{L^2(M,g)}} \|P_{\mathfrak{B}} u - v\|_{H(d)(M,g)} \\
&\qquad + \|F_h - F\|_{V_h^{k*}} + \left(\|Id - J_h\|_{\mathcal{L}(W_h^{k-1}, W_h^{k-1})} + \|Id - J_h\|_{\mathcal{L}(W_h^k, W_h^k)} \right) \|F\|_{H(d)\Omega^k(M,g)^*} \Bigg],
\end{aligned} \tag{3.15}$$

where $J_h := i_h^{k*} \circ i_h^k$, $i_h^{k*} : L^2\Omega^k(M,g) \to W_h^k$ is the L^2 adjoint of i_h^k, and $P_{\mathfrak{B}} : L^2\Omega^k(M,g) \to L^2\Omega^k(M,g)$ is the orthogonal projection onto the closed subspace $\mathfrak{B} = \{dv : v \in H(d)\Omega^{k-1}(M,g)\}$. The proof of Theorem 3.10 in [36] can be carried out with minimal modifications using $F \in H(d)\Omega^k(M,g)^*$, $F_h \in H(d)\Omega^k(M,g_h)^*$, and dual norms thereof in place of $\langle i_h^* f, \cdot\rangle_h$, $\langle f_h, \cdot\rangle_h$, and L^2-norms thereof, respectively; this is because, as noted in Remark 7, the weak forms are all well-posed for this more general class of functionals, and the dual norm can be used in place of the Cauchy-Schwarz inequality when obtaining the bound $\langle f_h - i_h^* f, v_h\rangle_h \leq \|f_h - i_h^* f\|_h \|v_h\|_{V_h}$. Again, this is not necessarily meaningful for the original strong formulation (3.11), but it can still be useful in practice.

Our main contribution that is not covered by existing literature will be putting bounds on $\|Id - J_h\|$ that depend only on intrinsic quantities associated to a Regge metric.

Theorem 9. *There exists C depending only on n and k such that, if $\|g-g_h\|_{L^\infty(M,g_h)} < \frac{1}{C}$, then*

$$\|Id - J_h\|_{\mathcal{L}(W_h^k, W_h^k)} \le \binom{n}{k} C \left[2 + C\|g_h - g\|_{L^\infty(M,g_h)}\right] \|g_h - g\|_{L^\infty(M,g_h)}, \tag{3.16}$$

where $\|g - g_h\|_{L^\infty(M,g_h)} = \operatorname{ess\,sup}_{x\in M} \sup_{V,W\in T_x(M)\setminus\{0\}} \frac{|(g-g_h)(V,W)|}{\|V\|_{g_h}\|W\|_{g_h}}$.

Remark 8. *Note that the h-dependent norms on the right-hand side of the above inequality can be replaced by h-independent norms using the fact that $\|g - g_h\|_{L^\infty(M,g_h)} \le C_{g_h,g}(M)^2\|g - g_h\|_{L^\infty(M,g)}$.*

Proof of theorem 9. J_h has an explicit formula in terms of the Hodge star operators for g and g_h. First, we note that, from the definition of the $L^2(M,g)$ and $L^2(M,g_h)$ inner products, we have that

$$\sum_{T\subseteq M} \int_T u \wedge \star_h(\star_h^{-1} \star v) = \int_M u\wedge \star v = \langle u, v\rangle_{L^2(M,g)} = \langle u, J_h v\rangle_{L^2(M,g_h)} = \sum_{T\subseteq M}\int_T u \wedge \star_h J_h v.$$

So, the two maps $v \mapsto \langle u, \star_h^{-1}\star v\rangle_{L^2(M,g_h)}$ and $v \mapsto \langle u, J_h v\rangle_{L^2(M,g_h)}$ are identical for any $u \in W_h^k$. This is only possible if $J_h = \star_h^{-1}\star$.

Firstly we will define some quantities related to any pair of orthonormal coframes. Let $\{\theta^i\}_{i=1}^n$ be a g-orthonormal coframe and $\{\theta_h^i\}_{i=1}^n$ be a g_h-orthonormal coframe, which induce the same orientation on $T_x^*(T)$. This gives us bases $\{\theta^I\}_{|I|=k}$ and $\{\theta_h^J\}_{|J|=k}$ for $\Lambda_x^k(M)$. They are related by the change of basis matrix (with each multi-index assigned an integer index) $\Theta_J^I := \langle \theta^I, \theta_h^J\rangle_h$. For each I there is also a unique multi-index called $[n]\setminus I$ and a quantity $\sigma(I)$ such that $\star\theta^I = (-1)^{\sigma(I)}\theta^{[n]\setminus I}$ and $\star_h\theta_h^I = (-1)^{\sigma(I)}\theta_h^{[n]\setminus I}$. We will use $\tilde{\Theta}$ to refer to the change of basis matrix for $\Lambda_x^{n-k}(M)$, i.e. $\tilde{\Theta}_L^K := \langle\theta^K, \theta_h^L\rangle_h$ for all multi-indices K and L, $|K| = |L| = n-k$. The enumeration of multi-indices can be selected so that the integer index corresponding to the multi-index $[n]\setminus I$ in $\tilde{\Theta}$ is the same as that for I in Θ. Since the dimension of $\Lambda_x^{n-k}(M)$ is the same as that of $\Lambda_x^k(M)$, the matrices Θ and $\tilde{\Theta}$ have the same shape.

Thus we can write:

$$\begin{aligned} J_h|_x\theta_h^I &= \star_h^{-1}\star\sum_K(\Theta^{-1})^I_K\theta^K = \star_h^{-1}\sum_K(-1)^{\sigma(K)}(\Theta^{-1})^I_K\theta^{[n]\setminus K} \\ &= \sum_{K,J}(-1)^{\sigma(K)+\sigma(J)}(\Theta^{-1})^I_K\tilde{\Theta}^{[n]\setminus K}_{[n]\setminus J}\theta_h^J. \end{aligned}$$

So, expressing the endomorphism $Id - J_h|_x$ as a matrix in the basis $\{\theta_h^I\}$, we get

$$(Id - J_h|_x)^I_J = \delta^I_J - (-1)^{\sigma(J)}\sum_K(-1)^{\sigma(K)}(\Theta^{-1})^I_K\tilde{\Theta}^{[n]\setminus K}_{[n]\setminus J}. \tag{3.17}$$

Now we pick a specific $\{\theta_h\}$ depending on $\{\theta\}$. Let the matrix G be defined by $(G^{-1})^i_j = \langle\theta^i, \theta^j\rangle_h$. Then let E be the Cholesky factor of G, so $\theta_h^i = \sum_j E^i_j\theta^j$ gives a g_h-orthonormal coframing. Then $\Theta^I_J = \langle(E^{-1})^{i_1}_l\theta^l_h \wedge \cdots \wedge (E^{-1})^{i_k}_m\theta^m_h, \theta^J_h\rangle_h = \det([(E^{-1})^{i_a}_{j_b}])$. Likewise, $\tilde{\Theta}^K_L = \det([E^{k_a}_{l_b}])$.

Since the Cholesky factorization, inverse, and determinant are all smooth functions, by Taylor series expansion there exists a constant C such that if $\|G - \delta\|_m < \frac{1}{C}$, then $\|\Theta^{-1} - \delta\|_m < C\|G - \delta\|_m$ and $\|\tilde{\Theta} - \delta\|_m \le C\|G - \delta\|_m$.

This constant C depends only on the derivatives of the determinant and Cholesky square root maps at the matrices $[\delta^{i_a}_{j_b}]$ and $[\delta^{([n]\setminus I)_a}_{([n]\setminus J)_b}]$ for each multi-index I and J, and the derivatives of the inverse map at the $\binom{n}{k} \times \binom{n}{k}$ and $k \times k$ identity matrices. Therefore it depends only on n and k.

Lastly, let $\tilde{\delta}^I_J := (-1)^{\sigma(J)}\delta^I_J$. Putting all this together with (3.17) we can conclude

$$\begin{aligned} |(Id - J_h|_x)^I_J| &= \left|\delta^I_J - (-1)^{\sigma(J)}\sum_K(\Theta^{-1}\tilde{\delta})^I_K\tilde{\Theta}^{[n]\setminus K}_{[n]\setminus J}\right| \\ &= \left|\delta^I_J - (-1)^{\sigma(J)}[\tilde{\delta} + \tilde{\delta}(\tilde{\Theta} - \delta) + (\Theta^{-1} - \delta)\tilde{\delta} + (\Theta^{-1} - \delta)\tilde{\delta}(\tilde{\Theta} - \delta)]^I_J\right| \\ &= \left|(-1)^{\sigma(J)}[\tilde{\delta}(\tilde{\Theta} - \delta) + (\Theta^{-1} - \delta)\tilde{\delta} + (\Theta^{-1} - \delta)\tilde{\delta}(\tilde{\Theta} - \delta)]^I_J\right| \\ &\le \operatorname*{ess\,sup}_{x\in M} C\,[2 + C\|G - \delta\|_m]\,\|G - \delta\|_m \\ &\le C\left[2 + C\|g - g_h\|_{L^\infty(M,g_h)}\right]\|g - g_h\|_{L^\infty(M,g_h)}. \end{aligned}$$

Since $Id - J_h$ extends to a bundle endomorphism $\Lambda^k(T) \to \Lambda^k(T)$ on each simplex $T \subseteq M$, it is clearly the case that

$$\|Id - J_h\|_{\mathcal{L}(W_h^k, W_h^k)} \le \operatorname*{ess\,sup}_{x \in M} \|Id - J_h|_x\|_2 \le \binom{n}{k} \operatorname*{ess\,sup}_{x \in M} \max_{I,J} |(Id - J_h|_x)^I_J|.$$

This gives the desired bound on $\|Id - J_h\|_{\mathcal{L}(W_h^k, W_h^k)}$. $\square$

Showing that a complex of finite element spaces $\{V_h^k\}_{k=0}^n$ and/or the approximated metric g_h actually have the approximation properties that lead to a convergent discretization would be a complicated matter, but we will sidestep it by assuming we are in a situation of Lemma 7: there exists a finite closed cover $\{W_i\}$ of M such that each triangle $T \subseteq M$ lies entirely within W_i for some i, and each W_i itself lies within a coordinate chart U_i, and the maps $f_{h,T}$ defining the triangles $T \subseteq M$ are affine when expressed in these coordinates. In practice, this is a convenient way to describe a computational mesh on a manifold (see remark 6 for an indication of how less-smooth coordinate maps could be used as well). For each $T \subseteq M$, we pick a specific $i(T)$ so that $T \subset W_{i(T)}$, and set $\|\cdot\|_E^2 := \sum_{T \subseteq M} \|\cdot\|^2_{L^2(T, \delta_{U_{i(T)}})}$.

Approximation properties of finite element spaces are usually proved for flat simplices in Euclidean space, which would correspond to using the norm $\|\cdot\|_E$. By Theorem 6, $\|\cdot\|_{L^2(M,g)} \le \sqrt{\binom{n}{k}} \max_i C_{\delta_{U_i},g}(W_i)^k \sqrt{D_{g,\delta_{U_i}}(W_i)} \|\cdot\|_E$. This constant does not depend on g_h, $\mathcal{T}_h$, or V_h, so approximation properties of V_h proved in the context of local coordinates are sufficient.

Likewise, $\|g - g_h\|_{L^\infty(M,g)} \le \max_i C_{\delta_{U_i},g}(W_i)^2 \|g - g_h\|_{L^\infty(W_i, \delta_{U_i})}$, so approximation properties of the metric g_h may be proved using local coordinates as well.

3.4 Calculation of Connection Forms as a Hodge-Laplace Problem

As an application of the geometric methods developed in the previous sections of this paper to solve a geometric problem, we will approximate the connection 1-form α associated to a special frame field (e_1, e_2) on a simply connected 2-dimensional manifold with a smooth Riemannian metric, called (M, g). Specifically, we wish to approximate $\alpha = A^1_2 := \langle \nabla e_2, e_1 \rangle$. We want the frame field to have the property that $\delta\alpha = 0$. In fact, we require an even stronger property, that $\alpha = \delta\omega$ for some 2-form ω.

First, we note that there actually does exist such a frame. Let (e_1, e_2) be *any* smooth orthonormal frame on (M, g) with corresponding connection form $B^1_2 = \beta = df + \delta\omega$, where the splitting is unique by the Hodge decomposition, and let $\Phi : M \to SO(2)$ be a map which rotates the frame (e_1, e_2) smoothly on M by the angle $\theta : M \to \mathbb{R}$. Then the frame $\Phi \cdot (e_1, e_2)$ has the connection form

$$A_\Phi{}^1_2 = (\Phi^{-1} d\Phi)^1_2 + \mathrm{Ad}(\Phi)(A)^1_2,$$

where Ad denotes the adjoint representation of $SO(2)$ on $\mathfrak{so}(2)$, defined by $\mathrm{Ad}(H)(w) := HwH^{-1}$.

Since M is 2-dimensional, Ad is the trivial action. Additionally, the $\mathfrak{so}(2)$-valued 1-form $\Phi^{-1}d\Phi$ is, in coordinates,

$$\begin{bmatrix} 0 & -d\theta \\ d\theta & 0 \end{bmatrix}.$$

Thus, if Φ is a rotation by $\theta = -f \mod 2\pi$, then $A_\Phi{}^1_2 = \delta\omega$, and $\delta A_\Phi{}^1_2 = \delta\delta\omega = 0$. The resulting frame $(\tilde{e_1}, \tilde{e_2})$ is also unique up to a constant rotation.

The condition we have imposed on A^1_2 is reminiscent of the Coulomb gauge from electrodynamics, where A would be the magnetic vector potential. While the physical significance of A^1_2 and the frame (e_1, e_2) are not obvious, the idea is basically the same as gauge fixing.

To set up a weak form of the equation for α, or more specifically $*\alpha$, we let $\omega = *u$, so that $\alpha = *du$. We then apply the fact that $d\alpha = KdA$, where dA is the volume form and K is the Gauss curvature. Thus, by applying integration by parts, we arrive at the weak form of our problem: find $u \in H(d)\Omega^0(M, g)$ such that, for all $v \in H(d)\Omega^0(M, g)$ and $c \in \ker d = \mathfrak{H}^0(M, g)$,

$$\langle du, dv\rangle_{L_2(M,g)} = \int_{\partial M} v\alpha - \int_M vKdA, \tag{3.18}$$

$$\langle u, c\rangle_{L^2(M,g)} = 0. \tag{3.19}$$

This is in fact a specialization of the abstract Hodge-Laplace problem for the $L^2(M, g)$-de Rham complex. Note that the right-hand side of (3.18) is zero when evaluated against any constant function $v \in \ker d$, meaning this equation is well-posed as written.

We will describe a discretization of the above problem. Let $\mathcal{T}_h$ be a Riemannian shape-regular and quasi-uniform family of triangulations of M (in the sense of Section 3.2.4) such that $h_T \leq h$ for all $T \in \mathcal{T}_h$, and let g_h be a Regge metric which is smooth on the interior of each simplex of $\mathcal{T}_h$ and approximates g. Using a finite-dimensional subspace $V_h \subset H(d)\Omega^0(M, g_h)$ such that for each $T \subseteq M$, $f^*_{h,T}(v_h) = \hat{v}_h$ for some $\hat{v}_h \in \hat{V} \subset C^2\Omega^0(\hat{T})$ for a fixed subspace $\hat{V}$, the discrete problem is: find $u_h \in V_h$ such that, for all $v_h \in V_h$ and $c \in V_h \cap \ker d$,

$$\langle du_h, dv_h\rangle_{L^2(M,g_h)} = \langle\langle \alpha|_{\partial M}{}_{\text{dist}}(g_h), v_h\rangle\rangle - \langle\langle KdA_{\text{dist}}(g_h), v_h\rangle\rangle \tag{3.20}$$

and

$$\langle u_h, c\rangle_{L^2(M,g_h)} = 0. \tag{3.21}$$

The functional KdA_{dist} above is the distributional Gaussian curvature, defined in [6, 28] as

$$\langle\langle KdA_{\text{dist}}(g_h), v_h\rangle\rangle := \sum_{T\subseteq M} \int_T v_h K_h dA_h + \sum_{\mathring{e}\subset \mathring{M}} \int_e v_h [\![k_h ds_h]\!] + \sum_{p\in \mathring{M}} v_h(p) \left(2\pi - \sum_{T\ni p} \theta_T(p)\right).$$

Above, k_h is the geodesic curvature of the edge e as measured by g_h and $\theta_T(p)$ is the interior angle of the triangle T at the point $p \in \partial T$ as measured by g_h. The functional

$\alpha|_{\partial M_{\text{dist}}}$ is defined as

$$\langle\langle \alpha|_{\partial M_{\text{dist}}}(g_h), v_h\rangle\rangle := \sum_{e\subset\partial M}\int_e v_h(d\mu - k_h ds_h) - \sum_{p\in\partial M} v_h(p)\left(\pi - [\![\mu]\!]|_p - \sum_{T\ni p}\theta_T(p)\right),$$

where μ is the angle that the smooth frame e_1 makes with the outward normal vector of ∂M as measured by the smooth metric g.

The above definition of $\alpha|_{\partial M_{\text{dist}}}$ is perhaps not very obvious. There are two ways to make sense of it; one involves proving a generalization of the Gauss-Bonnet theorem to manifolds with Regge metrics, as we have done in Section 2.4, and the other amounts to observing that the method converges with this definition. The computed form $\alpha_h = \star_h du_h$ may be interpreted as a form which approximates, in each triangle, the connection form associated to a special g_h-orthogonal frame field $(e_1, e_2)_h$ which makes the same angle (measured with g_h) with the outward normal vector as (e_1, e_2) does (measured with g) and such that its associated connection one-form is co-exact.

To take the second route, there is a generalization of Theorem 3.6 of [28] in the case $n = 2$:

Theorem 10. *Let v be a continuous function such that $v|_T \in C^2(T)$ for all $T \subseteq M$, let and $\{g(t) : t \in [0,1]\}$ be a family of Regge metrics such that $t \mapsto g(t)|_T$ is smooth for each triangle T. Then,*

$$\frac{\partial}{\partial t}[\langle\langle \alpha|_{\partial M_{\text{dist}}}(g(t)), v\rangle\rangle - \langle\langle K dA_{\text{dist}}(g(t)), v\rangle\rangle] = -\frac{1}{2}b_h(g(t); \dot g(t), v).$$

b_h is defined as

$$b_h(g;\sigma,v) := \sum_{T\subseteq M}\langle \mathbb{S}\sigma, \nabla\nabla v\rangle_{L^2(T,g)} - \sum_{e\subset M}\langle(\mathbb{S}\sigma)(n,n), dv([\![n]\!])\rangle_{L^2(e,g)}, \tag{3.22}$$

where $\mathbb{S}\sigma = \sigma - \operatorname{tr}(\sigma)g$ and n denotes the unit normal to e with respect to g.

Note that the definition of b_h only depends on the mesh $\mathcal{T}_h$ and the metric g. Note also that we abuse notation by using the letter n for the normal vector rather than the dimension (which in this context is fixed at 2).

Proof of Theorem 10. Since in our case $g = g(t)$, we will use a subscript t to emphasize the dependence on the $g(t)$ metric. Therefore

$$\begin{aligned}
\frac{\partial}{\partial t}\langle\langle \alpha|_{\partial M_{\text{dist}}}(g(t)), v\rangle\rangle &= \frac{\partial}{\partial t}\left[-\sum_{e\subset\partial M}\int_e v k_t ds_t - \sum_{p\in\partial M} v(p)\left(\pi - \sum_{T\ni p}\theta_T(p)\right)\right] \\
&= -\sum_{e\subset\partial M}\int_e v\frac{\partial}{\partial t}k_t ds_t + \sum_{p\in\partial M} v(p)\sum_{T\ni p}\frac{\partial}{\partial t}\theta_T(p) \\
&= -\frac{1}{2}\Bigg(\sum_{e\subset\partial M}\int_e v[d(\dot{g}(n_t,\tau_t))(\tau_t) + (\operatorname{div}(\mathbb{S}\dot{g}))(n_t)]ds_t \\
&\qquad - \sum_{p\in\partial M}\sum_{T\ni p} v(p)[\![\dot{g}(n_t,\tau_t)]\!]|_p\Bigg).
\end{aligned}$$

The identities used here are established in [28], Section 2. τ_t is a tangent unit vector to e.

In the second to last line of the proof of Lemma 3.3 of [28], the following identity is established (simplified here for the $n = 2$ case):

$$\begin{aligned}
b_h(g(t);\dot{g},v) = 2\frac{\partial}{\partial t}\langle\langle (KdA)_{\text{dist}}(g(t)), v\rangle\rangle &+ \sum_{p\in\partial M}\sum_{T\ni p} v(p)[\![\dot{g}(n_t,\tau_t)_T]\!]|_p \\
&- \sum_{e\subset\partial M}\int_e v[d(\dot{g}(n_t,\tau_t))(\tau_t) + (\operatorname{div}(\mathbb{S}\dot{g}))(n_t)]ds_t.
\end{aligned}$$

Comparing with the value of $\frac{\partial}{\partial t}\langle\langle \alpha|_{\partial M_{\text{dist}}}(g(t)), v\rangle\rangle$, we get the desired result. □

Define the functional $F_h(g(t)) \in V_h^*$ by $F_h(g(t))(v) := \langle\langle \alpha|_{\partial M_{\text{dist}}}(g(t)), v\rangle\rangle - \langle\langle KdA_{\text{dist}}(g(t)), v\rangle\rangle$. In the framework of geometric variational crimes, $F_h(g)$ corresponds to the functional F in (3.15) while $F_h(g_h)$ corresponds to the functional F_h on the right of the discrete problem (3.14) and in (3.15). Therefore a key component of the error analysis will be bounds on $\|F_h(g) - F_h(g_h)\|_{V_h^*}$.

3.5 Error Analysis of the Connection Form Problem

The purpose of this section is to establish asymptotic error bounds on $\|\star_h du_h - \star du\|_{L^2(M,g)}$ in terms of the metric error. We have already established in Theorem 10 that $\frac{\partial}{\partial t} F_h(g(t); v_h) = -\frac{1}{2} b_h(g(t); \dot{g}(t), v_h)$. In particular we can set $g(t) = tg_h + (1-t)g$, so that $\|F_h(g; \cdot) - F_h(g_h; \cdot)\|_{V_h^*} \leq \sup_{v_h \in V_h} \frac{\int_0^1 \frac{1}{2} |b_h(g(t); g_h - g, v_h)| dt}{\|v_h\|_{V_h}} \leq \sup_{t \in [0,1]} \frac{1}{2} \|b_h(g(t); g_h - g, \cdot)\|_{V_h^*}$.

Theorem 11. *Let (M, g) be a compact disc with Riemannian metric g, and let $\{(M, \mathcal{T}_h, g_h)\}_{h \in S}$ be a Riemannian shape-regular and quasi-uniform set of meshes and Regge metrics, and let $V_h \subset H(d)\Omega^0(M, g_h)$ be a space such that for each $v_h \in V_h$ and each triangle $T \subseteq M$, $v_h|_T = f_{h,T}^{-*}(\hat{v})$ for some $\hat{v} \in \hat{V} \subset C^2\Omega^0(\hat{T})$, where $\hat{V}$ is a fixed finite-dimensional subspace and $\hat{T}$ is the standard simplex. Assume additionally that there exists an integer $r \geq 0$ such that*

1. $\|g_h - g\|_{L^2(M,g)} = O(h^{r+1})$,
2. $\left(\sum_{T \subseteq M} \|\nabla g_h\|^2_{L^2(T,g)}\right)^{\frac{1}{2}} = O(h^r)$,
3. $\|g_h - g\|_{L^\infty(M,g)} \leq C' < 1$ *for all* $h \in S$,
4. $\max_{T \subseteq M} \|\hat{\nabla} g_h\|_{L^\infty(T,\hat{\delta})} + \|\hat{\nabla} g\|_{L^\infty(T,\hat{\delta})} = O(1)$, *where $\hat{\delta}$ is the flat metric induced by the map $f_{h,T} : \hat{T} \to T$ and $\hat{\nabla}$ is its associated Levi-Civita connection.*

Then, treating values that depend only on (M, g), C', and the mesh quality bounds K, K_V, and K' appearing in the definitions of shape-regularity and quasi-uniformity (see Section 3.2.4) as constants, we have

$$\|b_h(g(t); g - g_h, \cdot)\|_{V_h^*} \in O(h^r),$$

where b_h is as in equation (3.22).

Proof. We proceed by bounding each term of b_h separately. For clarity, we will elide values which are bounded uniformly by a constant that does not depend on h.

A few things need to be established to simplify the bounds. We'll show that $C_{g,g(t)}(M)$ and $C_{g(t),g}(M)$ both converge to 1 as fast as $\|g-g_h\|_{L^\infty(M,g)}$ converges to zero.

Since for any two Regge metrics g_1, g_2 which are both smooth in a neighborhood of $x \in M$,

$$\begin{aligned}
\sup_{V\in T_xM\setminus\{0\}} \frac{\|V\|_{g_1}^2}{\|V\|_{g_2}^2} &\leq \sup_{V,W\in T_xM\setminus\{0\}} \frac{\langle V,W\rangle_{g_1}}{\|V\|_{g_2}\|W\|_{g_2}} \\
&= \sup_{V,W\in T_xM\setminus\{0\}} \frac{\langle V,W\rangle_{g_1}-\langle V,W\rangle_{g_2}}{\|V\|_{g_2}\|W\|_{g_2}} + \frac{\langle V,W\rangle_{g_2}}{\|V\|_{g_2}\|W\|_{g_2}} \\
&\leq \|g_1-g_2\|_{L^\infty(M,g_2)} + 1,
\end{aligned}$$

we have $C_{g(t),g}(M)^2 \leq \|g(t)-g\|_{L^\infty(M,g)}+1$ and $C_{g,g(t)}(M)^2 \leq \|g(t)-g\|_{L^\infty(M,g(t))}+1$. Clearly, $\|g(t)-g\| \leq \|g_h-g\|$ in any metric. To make use of the second inequality, we need to do some work: $C_{g,g(t)}(M)^2 \leq \|g_h-g\|_{L^\infty(M,g(t))}+1 \leq C_{g,g(t)}(M)^2\|g_h-g\|_{L^\infty(M,g)}+1$, so we obtain $C_{g,g(t)}(M)^2 = 1+O(\|g_h-g\|_{L^\infty(M,g)})$. So, any instance of $C_{g(t),g}(T)$, $C_{g(t),g}(M)$, $C_{g,g(t)}(T)$, or $C_{g,g(t)}(M)$ will be suppressed. An identical argument allows us to suppress terms like $C_{g(t),g_h}$ and $C_{g_h,g(t)}$, as well as terms involving D (by part 1 of Lemma 6).

Another technicality that needs to be addressed: if $\{(M,\mathcal{T}_h,g_h)\}_{h\in S}$ is Riemannian shape-regular and quasi-uniform, then so is $\{(M,\mathcal{T}_h,tg+(1-t)g_h)\}_{h\in S}$ for any $t\in[0,1]$, with constants $K(t)$, $K_V(t)$, and $K'(t)$ bounded by a multiple of those for g_h. This is because $C_{\hat{\delta},g(t)}(T) \leq C_{\hat{\delta},g_h}(T)C_{g_h,g(t)}(T)$ and $C_{g(t),\hat{\delta}}(T) \leq C_{g(t),g_h}(T)C_{g_h,\hat{\delta}}(T)$.

Now let us begin bounding the terms of b_h. In what follows, we use the letter C to denote a constant that is independent of h and may change at each occurrence. We have

$$\begin{aligned}
|\langle \mathbb{S}_t\sigma, \nabla_t\nabla_t v_h\rangle_{L^2(T,g(t))}| &\leq \|\mathbb{S}_t\sigma\|_{L^2(T,g(t))}\|\nabla_t\nabla_t v_h\|_{L^2(T,g(t))} \\
&\leq C\|\sigma\|_{L^2(T,g(t))}h^{-1}\left(1+\|\hat{\nabla}g(t)\|_{L^\infty(T,\hat{\delta})}\right)\|dv_h\|_{L^2(T,g(t))} \\
&\leq C\|g-g_h\|_{L^2(T,g)}h^{-1}\left(1+\|\hat{\nabla}g_h\|_{L^\infty(T,\hat{\delta})}+\|\hat{\nabla}g\|_{L^\infty(T,\hat{\delta})}\right)\|dv_h\|_{L^2(T,g_h)}.
\end{aligned}$$

In the above derivation: the first line is an application of Cauchy-Schwarz, the second is applying the inverse inequality of corollary 4, and the last line is changing from $g(t)$

to g or g_h norms. Recall that $\mathbb{S}_t\sigma(V,W) = \sigma(V,W) - (\mathrm{Tr}_t\sigma)\langle V,W\rangle_t$, and $|\mathrm{Tr}_t\sigma| \leq 2\sup_{V,W\in T_x(T)} \frac{\sigma(V,W)}{\|V\|_t\|W\|_t}$, so $\|\mathbb{S}_t\sigma\|_{L^2(T,g(t))} \leq 3\|\sigma\|_{L^2(T,g(t))}$.

Next are the edge terms. Let n_α and n_β be the two outward-facing normal vectors of e for the two triangles $T_\alpha \supset e$ and $T_\beta \supset e$ in $g(t)|_{T_\alpha}$ and $g(t)|_{T_\beta}$ respectively. If $e \subset \partial M$, then we will say that $n_\beta = 0$ and the terms involving the non-existent T_β can be ignored. First we unravel the definitions:

$$\begin{aligned}
&|\langle(\mathbb{S}_t\sigma)(n_t,n_t), dv_h([\![n_t]\!]\rangle_{L^2(e,g(t))}| = |\langle\sigma(\tau_t,\tau_t), dv_h([\![n_t]\!])\rangle_{L^2(e,g(t))}| \\
&\quad\leq \|\sigma\|_{L^2(e,g(t))}\left(\|dv_h(n_\alpha)\|_{L^2(e,g(t))} + \|dv_h(n_\beta)\|_{L^2(e,g(t))}\right) \\
&\quad\leq \|\sigma\|_{L^2(e,g)}\left(\|dv_h\|_{L^2(e,g_h)} + \|dv_h\|_{L^2(e,g_h)}\right). \qquad (3.23)
\end{aligned}$$

Here we applied the Cauchy-Schwarz inequality and then converted from $g(t)$ norms to g and g_h norms. By a standard application of the scaled trace and inverse inequalities (3) and (4) to each coefficient of dv_h in the coordinates $(\hat{x},\hat{y})$ induced by the diffeomorphism $\hat{T} \to T$, we get that for each edge e and each $T_\alpha \supset e$,

$$\|dv_h\|_{L^2(e,g_h)} \leq Ch^{-\frac{1}{2}}\|dv_h\|_{L^2(T_\alpha,g_h)}.$$

Plugging this into (3.23), we get

$$\begin{aligned}
(3.23) &\leq C\|\sigma\|_{L^2(e,g)}h^{-\frac{1}{2}}\left(\|dv_h\|_{L^2(T_\alpha,g_h)} + \|dv_h\|_{L^2(T_\beta,g_h)}\right) \\
&\leq Ch^{-\frac{1}{2}}\left(h^{-\frac{1}{2}}(1+\|\hat{\nabla}g\|_{L^\infty(T_\alpha,\hat{\delta})})\|\sigma\|_{L^2(T_\alpha,g)} + h^{\frac{1}{2}}\|\nabla\sigma\|_{L^2(T_\alpha,g)}\right) \\
&\qquad\left(\|dv_h\|_{L^2(T_\alpha,g_h)} + \|dv_h\|_{L^2(T_\beta,g_h)}\right) \\
&= C\left(h^{-1}\left(1+\|\hat{\nabla}g\|_{L^\infty(T,\hat{\delta})}\right)\|g-g_h\|_{L^2(T_\alpha,g_h)} + \|\nabla g_h\|_{L^2(T_\alpha,g_h)}\right) \\
&\qquad\left(\|dv_h\|_{L^2(T_\alpha,g_h)} + \|dv_h\|_{L^2(T_\beta,g_h)}\right).
\end{aligned}$$

Here, we applied the trace inequality of Corollary 3 to the σ terms, then rearranged the h coefficients into a convenient form, then expanded the definition of σ. Note that $\nabla g = 0$.

From these two bounds on the different terms appearing in b_h, it is clear that

$$\begin{aligned}|b_h(g(t); g - g_h, v_h)| \leq C\|dv_h\|_{L^2(M,g_h)} \Bigg[\Bigg(\sum_{T \subset M} \|\nabla g_h\|^2_{L^2(T,g)} \Bigg)^{\frac{1}{2}} \\ + h^{-1}\|g - g_h\|_{L^2(M,g)} \left(2 + \max_{T \subseteq M} \|\hat{\nabla} g_h\|_{L^\infty(T,\hat{\delta})} + \|\hat{\nabla} g\|_{L^\infty(T,\hat{\delta})} \right) \Bigg] \\ \leq C\|dv_h\|_{L^2(M,g_h)} h^r,\end{aligned}$$

where C depends only on (M, g), the constant $C' < 1$ which bounds $\|g - g_h\|_{L^\infty(M,g)}$, and the mesh quality measures K, K_V, and K'. $\square$

Since $F_h(g; v_h) - F_h(g_h; v_h) = \frac{1}{2}\int_0^1 b_h(tg + (1-t)g_h; g - g_h, v_h)dt$, this immediately implies that $\|F_h(g; \cdot) - F_h(g_h; \cdot)\|_{V_h^*} = O(h^r)$.

With this, we can conclude that if we have a finite element space $V_h \subset H(d)\Omega^0(M, g_h)$ which satisfies the conditions of Theorem 11, and which is contained in a sequence of spaces admitting bounded cochain projections π_h^k, and which contains all constant functions and satisfies $\inf_{v \in V_h} \|u - v\|_{H(d)(M,g)} = O(h^r)$, and we have an interpolant g_h satisfying the hypotheses of Theorem 11 and $\|g - g_h\|_{L^\infty(M,g)} = O(h^r)$, then $\|du_h - du\|_{L^2(M,g)} = O(h^r)$ and hence $\| \star_h du_h - \star du\|_{L^2(M,g)} = O(h^r)$.

In the case that M has the topology of a 2-dimensional disc, it is practically sufficient to consider a shape-regular, quasi-uniform mesh $\mathcal{T}_h$ on a closed convex domain $\Omega \subset \mathbb{R}^2$ and construct a finite element space V_h using this mesh, such as Lagrange elements of order r. Additionally, setting $\bar{\delta}$ to the Euclidean metric induced by this coordinate choice, we have $\hat{\nabla} = \bar{\nabla}$ since $f_{h,T}$ is affine when expressed in this coordinate system, so $\|\hat{\nabla} g\|_{L^\infty(T,\hat{\delta})} \leq C_{\bar{\delta},\hat{\delta}}(T)^3 \|\bar{\nabla} g\|_{L^\infty(T,\bar{\delta})} \leq Ch^3 \|\bar{\nabla} g\|_{L^\infty(T,\bar{\delta})}$ if mesh elements have Euclidean diameter $O(h)$, and likewise for g_h. Order-r interpolants g_h for g satisfying the other hypotheses of Theorem 11 also exist, since interpolants with the correct convergence properties in Euclidean space exist [28] and the Euclidean metric is obviously related to the g metric in a bounded way.

3.6 Numerical Example and Benchmarking

The discrete problem presented in Section 4 was implemented in python using the NGSolve framework [54, 55]. In order to benchmark against an analytically solved example, such an example needed to be calculated. We chose a spherical cap of radius 1 with the parametrization $(x, y) \mapsto (x, y, \sqrt{1-x^2-y^2})$ because it is easy to calculate the Gauss curvature, and we were lucky to guess the correct frame:

$$e_1 = \sqrt{\frac{1-x^2-y^2}{1-y^2}}\frac{\partial}{\partial x},$$

$$e_2 = \frac{-xy}{\sqrt{1-y^2}}\frac{\partial}{\partial x} + \sqrt{1-y^2}\frac{\partial}{\partial y},$$

$$\alpha = -\frac{1}{\sqrt{1-x^2-y^2}}\left(ydx + \frac{xy^2}{1-y^2}dy\right).$$

It can be checked (with much labor) that $d\alpha = \frac{1}{\sqrt{1-x^2-y^2}}dx \wedge dy = KdA$, $d \star \alpha = 0$, and $\alpha = \langle \nabla e_2, e_1 \rangle$.

Our benchmarks were evaluated on the domain $(x, y) \in [-\frac{1}{4}, \frac{1}{4}]^2$. Meshes were created with the "GenerateMesh" function, generating unstructured meshes with (Euclidean) edge length approximately equal to h. Each interior vertex was then perturbed with uniform randomness in the range $[-\frac{h}{4}, \frac{h}{4}]^2$. The order-$r$ Regge finite element space [42] and interpolant described in [28, Appendix A] was used for g_h, and the order-r Lagrange finite element space was used for V_h. The python code used to produce this data can be provided upon request.

As one can see from Figure 1, the numerical scheme's convergence in the $L^2(M, g)$-norm very closely matches a priori predictions.

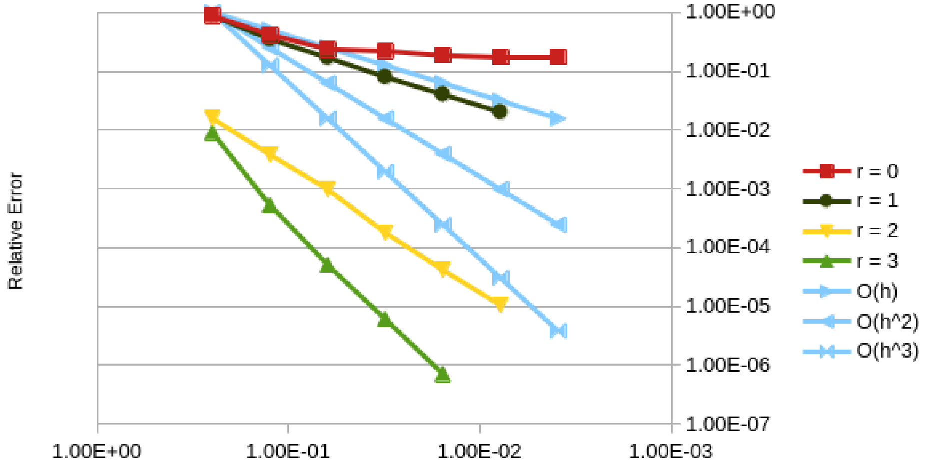

Figure 3.2: Graphs of the relative error $\|du_h - \star\alpha\|_{L^2(M,g)}/\|\alpha\|_{L^2(M,g)}$ vs. the parameter h which controls the diameter of mesh elements.

	$r = 0$		$r = 1$	
h	E	ndof	E	ndof
2.5E-1	8.75E-1	22	8.73E-1	66
1.3E-1	4.16E-1	86	3.48E-1	306
6.3E-2	2.36E-1	310	1.69E-1	1170
3.1E-2	2.18E-1	1286	7.93E-2	5010
1.6E-2	1.85E-1	4906	4.01E-2	19362
7.8E-3	1.73E-1	19334	2.00E-2	76818
3.9E-3	1.74E-1	76274		

	$r = 2$		$r = 3$	
h	E	ndof	E	ndof
2.5E-1	1.56E-2	134	9.00E-3	226
1.3E-1	3.80E-3	662	5.24E-4	1154
6.3E-2	9.75E-4	2582	5.06E-5	4546
3.1E-2	1.78E-4	11174	6.06E-6	19778
1.6E-2	4.15E-5	43370	7.09E-7	76930
7.8E-3	1.05E-5	172454		

Figure 3.3: Table containing h = average triangle diameter, E = relative error of du_h measured in the $L^2(M, g)$ norm, ndof = total degrees of freedom (including those used in constructing the Regge metric).

Chapter 4

Surface Tension in PISALE

4.1 Introduction

We now turn from abstract, intrinsic geometry to applied, extrinsic geometry.

Surface tension is the name given to the force which arises from the interface of a liquid with a gas or another liquid with which it cannot form a solution. It is very closely related to the *capillarity* or *adhesion* between a liquid and a solid. In all cases, the force arises from the *surface energy* of the interface. This energy is exactly the discrepancy between the electro-chemical potential energy of molecules that lie on the boundary of a liquid, and the potential energy of molecules internal to the liquid. The potential energy discrepancy per unit area of the interface is a number called γ, a function on S which depends on chemical properties of the interface, and the additional potential energy induced by the interface S is modeled as $\int_{\mathring{S}} \gamma dS$ [24].

By making some assumptions about the smoothness of the interface and applying the principle of virtual work (more details on this procedure will be provided in the next section), the stress term corresponding to this energy is [11]

$$\langle \Sigma_{\mathrm{st}}, V \rangle = \int_{\mathring{S}} (\gamma 2H\vec{n} + \nabla_S \gamma) \cdot V dS - \sum_{c} \int_{c} \sum_{\partial S_k \supset c} \gamma_k \epsilon_k \nu_k \cdot V dc, \tag{4.1}$$

where H is the mean curvature of S (measured with normal vector $\vec{n}$), $\vec{n}$ is the outward-pointing normal vector, $\nabla_S \gamma$ is the tangential derivative of γ along S, γ_k is the surface tension coefficient of the interface component S_k, c ranges over the set of regular intersections of interface components, and $\epsilon_k \vec{\nu}_k = \pm \vec{\nu}_k$ is the conormal vector of S_k at c (either inward-pointing or outward-pointing, depending on the orientation induced on c by S_k).

This could be interpreted as a force $F_{\mathrm{st}} = \gamma 2H\vec{n} + \nabla_S \gamma$ supported on S and a force $F_{\mathrm{cap}} = -\sum_k \gamma_k \epsilon_k \nu_k$ supported on the regular multi-points. These forces are singular, behaving like delta distributions on their support. Surface tension could thus be understood as a generalized hyperbolic mean curvature flow.

The different terms of this integral formula are usually treated differently in fluid mechanics literature. The term $\gamma 2H\vec{n}$ can be reinterpreted as a jump in the pressure values experienced by two fluids at the intersection, which is the content of the Young-Laplace equation. The $\nabla_S \gamma$ term is often referred to as the Marangoni force, which has been experimentally shown to result in unintuitive dynamics [13]. The terms at regular intersections between multiple interfaces are sometimes called capillary forces, and setting them to zero yields Young's relation on the contact angle [62].

Surface tension is a very important phenomenon to model for experiments relating to droplets and their shape. An application that was of interest for the Pacific Islands Structured Arbitrary Lagrangian Eulerian (PISALE) group was the dynamics of cryogenic hydrogen droplets heated by an x-ray free electron laser (XFEL) and how the expanding droplet interacts with unheated droplets as described by Parisuaña [4]. For this application, it was necessary to verify and update the surface tension module.

The currently used method for calculating surface tension forces uses *height functions*, which use the volume fractions stored in each computational cell to approximate the interface's as the graph of a function. A detailed look at these surface tension modules is given in the final section.

4.2 Geometrical Derivation of Surface Tension

In this section we will derive the surface tension force using the principle of virtual work. We begin with the assumption that S, the interface, is a finite union of oriented C^1 manifolds with corners, such that the intersection of any two distinct components S_k, S_j is a codimension-1 face of both. We also assume that the surface tension γ is a bounded integrable function defined on S which is independent of time and C^1 when restricted to the relative interior of each component.

Next we would like to apply the principle of virtual work. We will derive a force from an energy by using the definition

$$W_{\sigma|_{[t_1,t_2]}} = \int_{t_1}^{t_2} F \cdot \dot{\sigma} dt,$$

where σ is a hypothetical trajectory of the system and F is a force. For instance, $\sigma : \mathbb{R} \to \mathbb{R}^3$ may be the path a particle takes in 3-d space. So, if we assert that the alternate definition $-W_\sigma = E(t_2) - E(t_1)$ is also correct, where E is the total potential energy, we can derive that, for any trajectory σ,

$$F(\sigma(0)) \cdot \dot{\sigma}(0) = \lim_{h\to 0} \frac{1}{2h} \int_{-h}^{h} F \cdot \dot{\sigma} dt = -\frac{\partial}{\partial t}\Big|_{t=0} E(t).$$

In continuum mechanics, the configuration space is the set of diffeomorphisms $\mathbb{R}^3 \to \mathbb{R}^3$, and a trajectory is a smooth path through the space of such diffeomorphisms, in other words the flow of a vector field $V : \mathbb{R}^3 \to T\mathbb{R}^3$. E is the surface energy $E[S] = \int_{\mathring{S}} \gamma dS$. What the principle of virtual work tells us is that the surface tension force is the functional Σ_{st} defined by $\langle \Sigma_{\text{st}}, V\rangle = -\frac{\partial}{\partial t}|_{t=0} E[S(t)]$, where $S(t) = e^{Vt}(S)$ is the image of S under the time-t flow of the vector field V.

To calculate $\frac{\partial}{\partial t}|_{t=0} E[S(t)]$, first we will restrict ourselves to only one component S_k of S. Then take an oriented orthonormal frame $F(t) = (F_1(t), F_2(t), \vec{n}(t))$ for each $t \geq 0$, so that

$(F_1(t), F_2(t))$ is a positively oriented orthonormal frame on $S(t)$ and $dS(t) = \theta^1(t) \wedge \theta^2(t)$. Let all of these frames be extended to smooth orthonormal frames on a neighborhood Ω of S, so that $F(t) = F(0) \cdot U(t)$ for some map $U : \Omega \times [0,1] \to O(3)$, and set $\theta^i := \theta^i(0)$. Note that γ is transported along the flow of V as well; specifically, we will define $\gamma_k(t) := \gamma_k \circ e^{-Vt}$, so that $\gamma_k(t)$ is constant along any integral curve of V. We will also need to extend each $\gamma_k(t)$ to a smooth function on a neighborhood of S_k, with the special property that $\gamma_k(t)$ must have zero derivative in the direction normal to $S_k(t)$.

The form $\theta^1(t) \wedge \theta^2(t)$ is therefore equal to $(U_1^1 U_2^2 - U_1^2 U_2^1)\theta^1 \wedge \theta^2 + (U_1^1 U_2^3 - U_1^3 U_2^1)\theta^1 \wedge \theta^3 + (U_1^2 U_2^3 - U_1^3 U_2^2)\theta^2 \wedge \theta^3$. The coefficients are all determinants of 2x2 minors of U, so we will shorten it to $\det(A(t))\theta^1 \wedge \theta^2 + \det(B(t))\theta^1 \wedge \theta^3 + \det(C(t))\theta^2 \wedge \theta^3$. Note that $\det(A(0)) = 1$ and $\det(B(0)) = \det(C(0)) = 0$, since these are minors of the 3x3 identity matrix.

Therefore, because S_k is compact, the derivative can be moved into the integral sign:

$$\begin{aligned}
&\frac{\partial}{\partial t}|_{t=0} \int_{\mathring{S}_k(t)} i^*_{S_k(t)} \gamma_k(t) \theta^1(t) \wedge \theta^2(t) \\
&\quad= \frac{\partial}{\partial t}|_{t=0} \int_{\mathring{S}_k} i^*_{S_k} e^{Vt*} (\gamma_k(t) \theta^1(t) \wedge \theta^2(t)) \\
&\quad= \int_{\mathring{S}_k} \gamma_k i^*_{S_k} \left[\frac{\partial}{\partial t}|_{t=0} \det(A(t))\theta^1 \wedge \theta^2 + \frac{\partial}{\partial t}|_{t=0} \det(B(t))\theta^1 \wedge \theta^3 + \frac{\partial}{\partial t}|_{t=0} \det(C(t))\theta^2 \wedge \theta^3) \right] \\
&\qquad+ \int_{\mathring{S}_k} i^*_{S_k} \dot{\gamma}_k(0) \left[\det(A(0))\theta^1 \wedge \theta^2 + \det(B(0))\theta^1 \wedge \theta^3 + \det(C(0))\theta^2 \wedge \theta^3 \right] \\
&\qquad+ \int_{\mathring{S}_k} i^*_{S_k} \frac{\partial}{\partial t}|_{t=0} e^{Vt*} \left[\gamma_k \det(A(0))\theta^1 \wedge \theta^2 + \gamma_k \det(B(0))\theta^1 \wedge \theta^3 + \gamma_k \det(C(0))\theta^2 \wedge \theta^3 \right] \\
&\quad= \int_{\mathring{S}_k} \gamma_k \mathrm{tr}(\dot{A}(0))\theta^1 \wedge \theta^2 + \int_{\mathring{S}_k} i^*_{S_k} d\gamma_k(-V)\theta^1 \wedge \theta^2 + \int_{\mathring{S}_k} i^*_{S_k} L_V(\gamma_k \theta^1 \wedge \theta^2) \\
&\quad= \int_{\mathring{S}_k} -V \cdot \nabla_S \gamma_k dS + i^*_{S_k} L_V(\gamma_k \theta^1 \wedge \theta^2)
\end{aligned}$$

In the first line of the above derivation, the pullback formula was applied to $S_k(t) = e^{Vt}(S_k)$. In the second line the derivative was moved into the integral sign, $\theta^1(t) \wedge \theta^2(t)$ was expanded in terms of $\theta^i \wedge \theta^j$, and the chain and product rules were used to expand the derivative into three different terms (using the fact that $e^{V0} = Id$).

In the third line, the pullback was applied to the forms in the first term, canceling all but the one involving $\theta^1 \wedge \theta^2$, and the identity $\frac{\partial}{\partial t}|_{t=0} \det(A(t)) = \operatorname{tr}(\dot{A}(0))$ was applied. The derivative of $\gamma_k(t) = \gamma_k \circ e^{-Vt}$ was calculated explicitly in the second term. In the last term we used the known values of the determinants of $A(0)$, $B(0)$, and $C(0)$ and used the definition $L_V(\alpha) := \frac{\partial}{\partial t}|_{t=0} e^{Vt^*}\alpha$.

In the fourth line, we applied the fact that A is the upper 2x2 submatrix of the $O(n)$-valued matrix field U, so that $\dot{A}(0) \in \mathfrak{so}(2)$ and its trace is zero.

To simplify the Lie derivative term, we will use "Cartan's magic formula", which is the identity $L_V(\alpha) = V \lrcorner d\alpha + d(V \lrcorner \alpha)$. So, we get:

$$\begin{aligned}
&\int_{\mathring{S}_k} i^*_{S_k} \left(V \lrcorner d(\gamma_k \theta^1 \wedge \theta^2) + d(V \lrcorner \gamma_k \theta^1 \wedge \theta^2)\right) \\
&= \int_{\mathring{S}_k} i^*_{S_k} \left(V \lrcorner (d\gamma_k \wedge \theta^1 \wedge \theta^2 + \gamma_k d\theta^1 \wedge \theta^2 - \gamma_k \theta^1 \wedge d\theta^2)\right) + \int_{\partial S_k \setminus E_k} i^*_{\partial S_k} (V \lrcorner \gamma_k \theta^1 \wedge \theta^1) \\
&= \int_{\mathring{S}_k} \gamma_k i^*_{S_k} V \lrcorner (-\omega^1_j \wedge \theta^j \wedge \theta^2 + \theta^1 \wedge \omega^2_j \wedge \theta^j) + \int_{\partial S_k \setminus E_k} \gamma_k V \cdot \nu_k dc_k.
\end{aligned}$$

In the first line above, the integration by parts theorem for Whitney manifolds was applied and the exterior derivatives were expanded out. In the second line, the structure equations $d\theta^i = -\omega^i_j \wedge \theta^j$ were applied, where $\omega^i_j(X) := \langle \nabla_X F_j, F_i \rangle$ is the connection form, and we applied the fact that γ_k does not vary in the normal direction, so that $d\gamma_k \in \operatorname{Span}(\{\theta^1, \theta^2\})$. Finally we used the fact that $i^*_{\partial S_k}(V \lrcorner \theta^1 \wedge \theta^2) = V \cdot \nu_k dc_k$, where dc_k is the length form on $\partial S_k \setminus E_k$ induced by the orientation of S_k and ν_k is the outward conormal vector.

Note that $\omega^i_i = 0$ for any i, and define Γ^i_{jl} by $\omega^i_j = \Gamma^i_{jl} \theta^l$. Then we can simplify this integral yet further:

$$\begin{aligned}
&\int_{\mathring{S}_k} \gamma_k i^*_{S_k}\left(V \lrcorner(-\omega^1_j \wedge \theta^j \wedge \theta^2 + \theta^1 \wedge \omega^2_j \wedge \theta^j)\right)\\
&= \int_{\mathring{S}_k} \gamma_k i^*_{S_k}\left(V \lrcorner(-\Gamma^1_{31}\theta^1 \wedge \theta^3 \wedge \theta^2 + \Gamma^2_{32}\theta^1 \wedge \theta^2 \wedge \theta^3)\right)\\
&= \int_{\mathring{S}_k} \gamma_k i^*_{S_k}\left(V \lrcorner(\Gamma^1_{31} + \Gamma^2_{32})\theta^1 \wedge \theta^2 \wedge \theta^3\right)\\
&= \int_{\mathring{S}_k} \gamma_k (\Gamma^1_{31} + \Gamma^2_{32})\theta^3(V)\theta^1 \wedge \theta^2
\end{aligned}$$

Lastly, note that $\Gamma^1_{31} + \Gamma^2_{32} = -\Gamma^3_{11} - \Gamma^3_{22} = -\langle \nabla_{F_1}\vec{n}, F_1\rangle - \langle \nabla_{F_2}\vec{n}, F_2\rangle = -2H$ where H is the mean curvature, which is defined as one half the trace of the second fundamental form.

In summary, we have that the variation of the energy of the k'th component is equal to

$$\frac{\partial}{\partial t}|_{t=0} E[S_k(t)] = \int_{\mathring{S}_k} (-2H\gamma_k\vec{n} - \nabla_{S_k}\gamma_k) \cdot V dS + \int_{\partial S_k \setminus E_k} \gamma_k V \cdot \nu_k dc_k.$$

When all these terms are added together, we must take care to track orientations on ∂S_k, since we can only choose one orientation $dc = \pm dc_k$ for each of the arcs c that lie at the intersection of two or more interface components. Nonetheless, we get the claimed equation (4.1):

$$\langle \Sigma_{\mathrm{st}}, V\rangle := -\frac{\partial}{\partial t}|_{t=0} E[S(t)] = \int_{\mathring{S}} (2H\gamma\vec{n} + \nabla_S\gamma) \cdot V dS - \sum_c \int_c \sum_{\partial S_k \supset c} \gamma_k \epsilon_k \nu_k \cdot V dc.$$

4.3 PISALE

PISALE uses a staggered-grid, Lagrangian formulation with position and velocity being nodal variables and density, internal energy, temperature, pressure, strain, and stress being zonal (cell centered) variables [40]. The mesh is regular and (topologically) cubical, but the nodes may change position over time (this is what 'Lagrangian' means). In this Lagrangian formulation (in vector and indical notation $i, j, k = 1, 2, 3$) we have:

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\vec{U} = \rho U_{i,i} \tag{4.2}$$

$$\frac{D\vec{U}}{Dt} = \frac{1}{\rho}(\nabla\cdot\boldsymbol{\sigma} + B) = \frac{1}{\rho}(\sigma_{ij,j} + B_i) \tag{4.3}$$

$$\frac{De}{Dt} = \frac{1}{\rho}V\boldsymbol{s} : \dot{\boldsymbol{\varepsilon}} - P\dot{V} = \frac{1}{\rho}V\left(s_{ij}\dot{\varepsilon}_{ij}\right) - P\dot{V} \tag{4.4}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{U}\cdot\nabla$$

is the substantial derivative, ρ is the density, $\vec{U} = (u, v, w)$ is the material velocity, B is the body force per unit volume, t is time, $\boldsymbol{\sigma}$ is the total stress tensor, P is the pressure, e is the internal energy, V is the relative volume ($\rho V = \rho_0$ where ρ_0 is the reference density), $\boldsymbol{s}$ is the deviatoric stress defined as

$$s_{ij} = \sigma_{ij} + P\delta_{ij}, \tag{4.5}$$

where δ is the Kronecker delta, and $\dot{\boldsymbol{\varepsilon}}$ is the strain rate tensor defined as

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right). \tag{4.6}$$

These equations are discretized using finite volumes for advection and finite differences to calculate stress. A Volume Of Fluid (VOF) formulation with interface reconstruction is used to represent multiple materials and handle material advection during the ALE remap step. During a simulation, the volume fraction of each material in a mixed zone is stored and the actual interface is only reconstructed as needed, such as during mesh refinement. Volume fractions in mixed zones are used for weighting the pressures, densities, stresses, etc., to obtain cell averaged quantities. Volume fractions are used during refinement of neighboring zones to determine the orientation of each interface and the actual location is determined by the volume fraction in the zone. A 2nd-order predictor-corrector model is utilized for time integration. We mix the ALE scheme with adaptive mesh refinement

(AMR) and in the code both the ALE and the AMR schemes can work independently to give zonal modification and potentially improve accuracy and/or runtime to solution.

4.4 Surface Tension Approximation

Surface tension models within the PISALE framework have been implemented in several ways [44]. These models were originally introduced to allow for simulation of phenomena such as droplet deformation in Extreme Ultraviolet(EUV) lithography [51], and include the ability to deal with multiple material interfaces unlike more restrictive methods based on level sets [57]. One method is implemented by adding a third-order space derivative to the stress that is derived thermodynamically based on the Van der Waals-Cahn-Hilliard free energy. However, this is no longer used because it required excessively small mesh sizes to accurately resolve the boundary.

A second approach is based on two-fluid Volume-of-Fluid (VOF) models, based on the so-called Continuous Surface Force (CSF) method [10]. In this method, the interface is approximated with a region of finite thickness $\delta > 0$, where δ is approximately the cell width, composed of level sets of a smoothed indicator function $\tilde{c}$, which (outside of the interface region) is equal to 1 at the interior of one fluid and equal to 0 at the interior of another. The surface tension stress term F_{st} is approximated using a body force which is equal to the total force induced by all level sets in the interface, divided by δ to approximate the singular nature of these body forces. The capillary term F_{cap} is neglected in this formulation.

The CSF method requires three main components: an approximation of $\tilde{c}$, an approximation of the normal vector $\vec{n} = \frac{\nabla\tilde{c}}{\|\nabla\tilde{c}\|}$, and an approximation of the curvature $2H \approx \nabla \cdot \vec{n}$. Brackbill's original implementation of the CSF method simply discretized $2H \approx \nabla \cdot \frac{\nabla\tilde{c}}{\|\nabla\tilde{c}\|}$ directly from a piecewise-constant VOF field $\tilde{c}$ using finite differences, but this results in so-called *parasitic currents* which are organized, persistent miscalculations of the surface tension force that ultimately deform droplets to an unacceptable degree. Much work has gone into removing parasitic currents from CSF-based methods [19, 35, 45].

A popular way to smooth $\tilde{c}$ that can help with parasitic currents is to use *height functions*. These work by first choosing a *stencil* for each cell C, which is an oriented rectangle of cells centered at C, oriented in the sense that one of the logical mesh coordinates i, j, k is "up". In the case of PISALE, this is a $3x3x7$ rectangle in 3d. Assuming the "up" direction is the k coordinate, the height function is a map F on the other logical coordinates, given by $F(i, j) = \sum_k \tilde{c}(i, j, k)$. A typical method, implemented in PISALE, is to take a polynomial approximation P of F using a least-squares fit, and set $\vec{n}_L$ to the normal vector of the level set $P = \frac{1}{2}$. Then a linear approximation ψ of the (implicit) coordinate map $\Psi(i, j, k) = (x, y, z)$ which maps logical coordinates to physical coordinates is calculated using central differences, and the approximate normal vector $\vec{n} = \psi(\vec{n}_L)$ and the corresponding approximate mean curvature H at the cell center C can be calculated and redistributed to the cell's vertices in a straightforward manner using only ψ and P.

Classical height functions perform well when the coordinate map Ψ is linear in a neighborhood of C, so that ψ is exact. In this case the discretization error is $O(h^2)$, where h is the average cell width. However, for even modest nonlinearities, it is possible for the method to fail to converge. Since Ψ is generally only bilinear on the interior of each cell, this is quite bad, especially when the mesh is being allowed to evolve over time.

In two dimensions, an alternative method is available. Instead of using the height function directly to calculate a normal vector and mean curvature, the height function can be used to select three sample points. Specifically, in each of the three columns $i = -1, 0, 1$, the height function $F(i)$ selects a cell $C_i = (i, \lfloor F(i) \rfloor)$ and a fraction $r_i = F(i) - \lfloor F(i) \rfloor$ within that cell. The fraction r_i is used to pick a point in the quadrilateral $\Psi(C_i)$ with vertices $v_{11}, v_{12}, v_{21}, v_{22}$, by $x_i = \frac{r_i}{2}(v_{11} + v_{12}) + \frac{1 - r_i}{2}(v_{21} + v_{22})$. The three points x_{-1}, x_0, x_1 lie in a unique circle, whose radius of curvature can be calculated directly by inverting a $2x2$ matrix. This method was implemented and evaluated extensively in PISALE [47], displaying first-order convergence even on some meshes with strong nonlinearities.

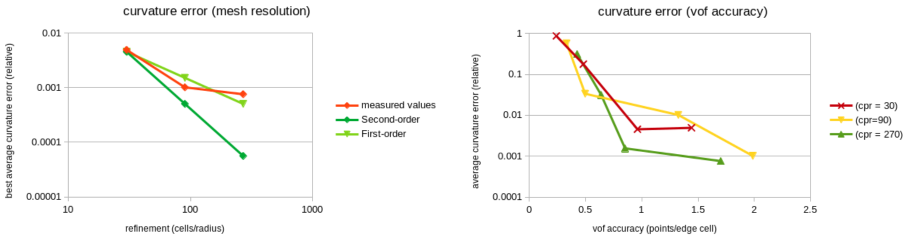

Figure 4.1: Plots of relative error in the estimated curvature of a circle measured at different refinement levels (cpr = cells per radius) for the osculating circle method, showing a clear first-order convergence in the curvature with respect to mesh refinement (left) and higher order convergence with respect to improvement of the VOF field (right), which is measured by the number of sides used to approximate a circle with a polygon when constructing the volume of fluid field.

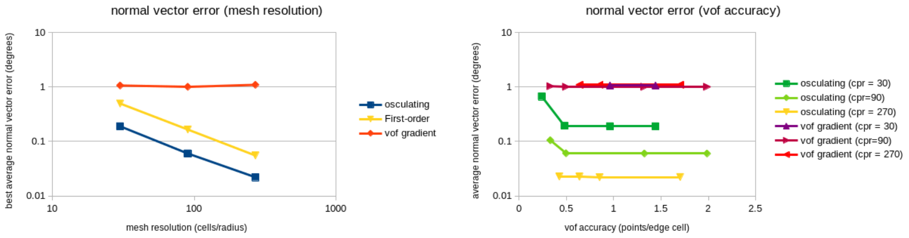

Figure 4.2: Plots of absolute error in the estimated normal vector of a circle for both the gradient-based method and the osculating circle based method, showing the osculating circles method has first-order convergence with respect to mesh width (left) while the gradient-based method does not, and neither method has convergence strongly dependent on the volume of fluid field accuracy (right).

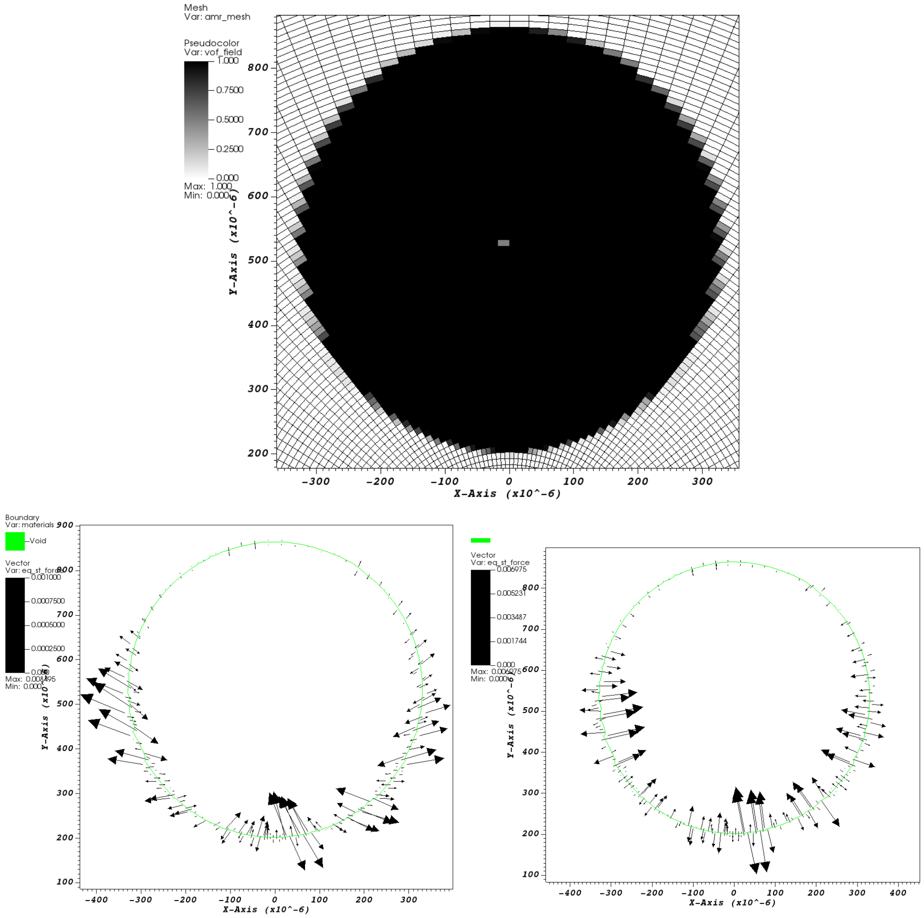

Figure 4.3: Top: Simulation setup for a non-conformal, nonlinear mesh test, approximating the curvature and normal vector of a circular droplet suspended in vacuum. Cells colored black are inside the droplet while cells colored white are outside the droplet.

Bottom Left: Approximate curvature vectors using polynomial interpolation for curvature and gradients for the normal vector.

Bottom Right: Approximate curvature vectors using osculating circle interpolation for both curvature and the normal vector.

Chapter 5

Appendices

5.1 Appendix A: Blow-Ups

Here we will provide a rapid overview of some key concepts that we have used, some of which are nonstandard. The primary reference for the term "blow-up" as we have used it is [5], although the concept was introduced earlier in at least [48].

What we will call a *Blow-Up* of a compact polyhedral manifold, called B_M, is any compact polyhedral manifold of the same dimension as M possessing a continuous onto map of polyhedral manifolds $\Phi : B_M \to M$ such that:

1. $\Phi|_{\mathring{B}_M} : \mathring{B}_M \to \mathring{M}$ is a diffeomorphism.
2. For all $d > 0$, the preimage of the relative interior of a codimension-d face Δ_d of M is the relative interior of a codimension-1 face $\hat{\Delta}_d$ of B_M, and all codimension-1 faces of B_M are obtained this way.
3. $\Phi|_{\Phi^{-1}(\mathring{e})} : \Phi^{-1}(\mathring{e}) \to \mathring{e}$ is a diffeomorphism for each codimension-1 face $e \subset \partial M$.
4. $\Phi|_{\Phi^{-1}(\mathring{\Delta}_d)} : \Phi^{-1}(\mathring{\Delta}_d) \to \mathring{\Delta}_d$ is a smooth submersion for each codimension-d face $\Delta_d \subset \partial M$, $0 < d \leq n$.

5. Two codimension-1 faces $\hat{\Delta}_d, \hat{\Delta}_{d'}$ of B_M have non-empty intersection if and only if $\Delta_d \subseteq \Delta_{d'}$ or $\Delta_{d'} \subseteq \Delta_d$.

If M is oriented, then we also require that B_M is oriented and Φ is an orientation-preserving map.

The easiest example of a polyhedral manifold with a blow-up is the standard unit n-simplex. Intuitively, the blow-up is the result of intersecting a new half-hyperplane for each d-face, $0 \leq d < n$, essentially "cutting off" corners, and defining the map Φ by contracting the newly created faces to the original d-faces. The process of cutting off corners is known as omnitruncation, and for a simplex it produces a permutahedron.

Actually writing down an explicit blow-down map between polytopes seems to be quite difficult; we are aware of [46] where a map from the blown-up simplex (the permutahedron) to the simplex is constructed that has all the needed properties, although only its inverse can be written down explicitly.

5.2 Appendix B: Relationship with Existing Formulas

The following proposition shows that the distributional curvature defined in (2.22) is equivalent to the equation for the densitized distributional curvature defined by other authors, with a slightly different test space.

Proposition 1. *Let f be a compatible frame and let $\hat{\phi}$ be a skew-symmetric, compatible* $\mathrm{End}(TM)$*-valued $(n-2)$-form. Then we can define a $(0,4)$-tensor $A(X,Y,Z,W) := (-1)^n \langle \star\phi(X,Y)W, Z\rangle$. Then, using notation from this paper on the left side and notation from [32] on the right side, the following are true at each point of $\mathring{T}$, $\mathring{\mathrm{e}}$, and $\mathring{\mathrm{p}}$ respectively:*

$$\langle \hat{R} \wedge \hat{\phi} \rangle = \frac{1}{2} \langle \mathcal{R}, A \rangle \omega_T, \tag{5.1}$$

$$\langle \hat{\mathrm{I\!I}}_e^T \wedge i_{\mathring{e}}^* \hat{\phi}|_T \rangle = -2 \langle \mathrm{I\!I}^{\vec{\nu}^T}, A_{F\hat{\nu}\hat{\nu}F} \rangle \omega_e^T, \tag{5.2}$$

$$\langle \hat{\Theta}_p^T \wedge i_{\mathring{p}}^* \hat{\phi}^T \rangle = 2 \Theta_p A_{\hat{\mu}\hat{\nu}\hat{\nu}\hat{\mu}} \omega_p. \tag{5.3}$$

In addition, $A_{F\hat{\nu}\hat{\nu}F}$ *is well-defined on* $\mathring{e}$*. To be specific,* ω_e^T *is the induced volume form from the orientation of* T *on* e*, and* ω_p *is the induced volume form from the orientation of* e *on* T *where* e *is the side such that* $E_{p,e}^T$ *has an inwards-pointing normal vector.*

Proof. The proof strategy for all three statements is to express the left-hand side in an appropriate basis and compute. For convenience, we'll use upper indices to refer to the coframe of a corresponding frame. So for instance, $\{E_e^i\}_{i=1}^n$ is the coframe on e defined by $E_e^i(E_{ej}) = \delta_j^i$.

To prove (5.1) we let ${\phi^i}_{jkl} := (-1)^n \langle \star\hat{\phi}(f_k, f_l) f_j, f_i\rangle = A(f_k, f_l, f_i, f_j)$, so $\hat{\phi} = \frac{1}{2}{\phi^i}_{jkl} f_i \otimes f^j \otimes \star(f^k \wedge f^l)$, and also write $\hat{R} = \frac{1}{2}\hat{R}(f_c, f_d) \otimes (f^c \wedge f^d)$. Note that these are not basis expansions for these forms, for instance the expression for $\hat{\phi}$ includes a $f_i \otimes f^j \otimes \star(f^k \wedge f^l)$ term and a $f_i \otimes f^j \otimes \star(f^l \wedge f^k)$ term for each i, j, k, l (hence the $\frac{1}{2}$ factor). Then we compute:

$$\begin{aligned}
\langle \hat{R} \wedge \hat{\phi}\rangle &= \frac{1}{4}\langle \hat{R}(f_c, f_d), {\phi^i}_{jkl} f_i \otimes f^j\rangle f^c \wedge f^d \wedge \star(f^k \wedge f^l) \\
&= \frac{1}{4}\sum_{k,l}\Big[\langle \hat{R}(f_k, f_l), {\phi^i}_{jkl} f_i \otimes f^j\rangle - \langle \hat{R}(f_l, f_k), {\phi^i}_{jkl} f_i \otimes f^j\rangle\Big]\omega_T \\
&= \frac{1}{2}\sum_{j,k,l}\langle \hat{R}(f_k, f_l) f_i, f_j\rangle_T {\phi^i}_{jkl}\omega_T \\
&= \frac{1}{2}\sum_{i,j,k,l}\langle \hat{R}(f_k, f_l) f_i, f_j\rangle_T A(f_k, f_l, f_i, f_j)\omega_T = \frac{1}{2}\langle \mathcal{R}, A\rangle \omega_T.
\end{aligned}$$

To prove (5.2), we will use the shorthand $E_i^T = (E_e^T)_i$, where E_e^T is the orthonormal frame adapted to e using notation from this paper. Without the superscript T, E_i is implicitly one of the first $n-1$ entries of E_e^T which do not depend on T. Let ${\phi^i}_{jkl} := (-1)^n \langle \star_T \hat{\phi}|_T(E_k^T, E_l^T) E_j^T, E_i^T\rangle_T = A|_T(E_k^T, E_l^T, E_i^T, E_j^T)$ so again $\hat{\phi} = \frac{1}{2}{\phi^i}_{jkl} E_i^T \otimes {E^j}^T \otimes \star_T({E^k}^T \wedge {E^l}^T)$. Then we compute:

$$\langle \hat{\mathbb{I}}_e^T \wedge i_{\mathring{e}}^* \hat{\phi}|_T\rangle = \frac{1}{2}\langle \hat{\mathbb{I}}_e^T(E_m), {\phi^i}_{jkl} E_i^T \otimes {E^j}^T\rangle E^m \wedge i_{\mathring{e}}^* \star_T ({E^k}^T \wedge {E^l}^T).$$

Here we need to use the fact that

$$i_{\mathring{e}}^* \star_T (E^{k^T} \wedge E^{l^T}) = \begin{cases} 0 & \text{if } k, l \neq n, \\ - \star_e E^{k^T} & \text{if } l = n, k \neq n \\ \star_e E^{l^T} & \text{if } k = n, l \neq n \end{cases}$$

where $\star_e$ means the Hodge star operator in the codimension-1 polytope e with the orientation induced by the orientation of T. This can be verified using the formula for the Hodge star in an orthonormal coframe along with the fact that the volume form induced on e from the orientation on T is $\omega_e^T = i_{\mathring{e}}^*(E_n^T \lrcorner \omega_T)$. Then the previous line simplifies to

$$\begin{aligned}
&\frac{1}{2} \langle \hat{\mathrm{I\!I}}_e^T(E_m), \phi^i{}_{jkl} E_i^T \otimes E^{j^T} \rangle E^m \wedge i_{\mathring{e}}^* \star_T (E^{k^T} \wedge E^{l^T}) \\
&\quad = -\frac{1}{2} \left[\sum_k \langle \hat{\mathrm{I\!I}}_e^T(E_k), \phi^i{}_{jkn} E_i^T \otimes E^{j^T} \rangle - \sum_l \langle \hat{\mathrm{I\!I}}_e^T(E_l), \phi^i{}_{jnl} E_i^T \otimes E^{j^T} \rangle \right] \omega_e^T \\
&\quad = -\sum_k \langle \hat{\mathrm{I\!I}}_e^T(E_k), \phi^i{}_{jkn} E_i^T \otimes E^{j^T} \rangle \omega_e^T \\
&\quad = -\sum_{j,k} \langle \hat{\mathrm{I\!I}}_e^T(E_k) E_i^T, E_j^T \rangle_T \phi^i{}_{jkn} \omega_e^T.
\end{aligned}$$

Now we use the definition that

$$\langle \hat{\mathrm{I\!I}}_e^T(E_k) E_i^T, E_j^T \rangle_T := \begin{cases} 0 & \text{if } i, j \neq n, \\ \langle \nabla_{E_k}^T E_i^T, E_j^T \rangle_T & \text{if } i = n \text{ or } j = n, \end{cases}$$

and abandon Einstein notation to avoid confusion about the n indices, so the previous line is equal to

$$
\begin{aligned}
-\sum_{i,j,k}&\langle \hat{\mathrm{I\!I}}_e^T(E_k)E_i^T, E_j^T\rangle_T \phi^i{}_{jkn}\omega_e^T \\
&= -\sum_{j,k}\big[\langle \nabla^T_{E_k}E_n^T, E_j\rangle_T \phi^n{}_{jkn} + \sum_{i,k}\langle \nabla^T_{E_k}E_i, E_n^T\rangle_T \phi^i{}_{nkn}\big]\omega_e^T \\
&= -\sum_{j,k}\big[-\langle \nabla^T_{E_k}E_n^T, E_j\rangle_T \phi^j{}_{nkn} + \langle \nabla^T_{E_k}E_j, E_n^T\rangle_T \phi^j{}_{nkn}\big]\omega_e^T \\
&= -2\sum_{j,k}\langle \nabla^T_{E_k}E_j, E_n^T\rangle_T \phi^j{}_{nkn}\omega_e^T \\
&= 2\sum_{j,k}\mathrm{I\!I}^{\vec{\nu}^T}(E_k, E_j)A|_T(E_k, E_n^T, E_j, E_n^T)\omega_e^T = -2\langle \mathrm{I\!I}^{\vec{\nu}^T}, A_{F\hat{\nu}\hat{\nu}F}\rangle\omega_e^T.
\end{aligned}
$$

(Recall that in [32], $\vec{\nu}^T$ is the inward-pointing normal vector, whereas E_n^T is the outward-pointing normal vector.)

We still need to show that $A_{F\hat{\nu}\hat{\nu}F}$ is well-defined. Assume that T is the side of $e = T \cap T'$ such that E_e^T is positively oriented. Then we have

$$
\begin{aligned}
A_{F\hat{\nu}\hat{\nu}F}(E_i, E_j)|_T &= (-1)^n\langle \star_T\hat{\phi}|_T(E_i, E_n^T)E_j, E_n^T\rangle_T \\
&= (-1)^{i-1}\langle \hat{\phi}|_T(E_1, \dots, \hat{E}_i, \dots E_{n-1})E_j, E_n^T\rangle_T \\
&= (-1)^{i-1}[i_{\mathring{e}}^*\hat{\phi}|_T(E_1, \dots, \hat{E}_i, \dots, E_{n-1})]_{E_e^T}{}^n{}_j \\
&= (-1)^{i-1}\mathrm{Ad}\left(\begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix}\right)\left([i_{\mathring{e}}^*\hat{\phi}|_{T'}(E_1, \dots, \hat{E}_i, \dots, E_{n-1})]_{E_e^{T'}}\right)^n{}_j \\
&= (-1)^i[i_{\mathring{e}}^*\hat{\phi}|_{T'}(E_1, \dots, \hat{E}_i, \dots, E_{n-1})]_{E_e^{T'}}{}^n{}_j \\
&= (-1)^i\langle \hat{\phi}|_{T'}(E_1, \dots, \hat{E}_i, \dots, E_{n-1})E_j, E_n^{T'}\rangle_{T'}.
\end{aligned}
$$

The notation $\hat{E}_i$ above means the ith vector is skipped. Here we need to apply the fact that the frame $E_e^{T'}$ is negatively oriented on M, so that $\star_{T'}\hat{\phi}|_{T'}(E_i, E_n^{T'}) = (-1)^{n+i}\hat{\phi}|_{T'}(E_1, \dots, \hat{E}_i, \dots, E_{n-1})$. This shows that $A_{F\hat{\nu}\hat{\nu}F}|_{T'} = A_{F\hat{\nu}\hat{\nu}F}|_T$.

To prove (5.3), we will use the shorthand $\tau_i^T = (E_{p,e}^T)_i$, where $E_{p,e}^T$ is the positively oriented orthonormal frame adapted to p using notation from this paper, and $e \supset p$ is the facet for which $\tau_n = -\vec{n}^T$. Without the superscript T, τ_i is implicitly one of the first $n-2$ entries of $E^T_{p,e}$ which do not depend on e or T. Let $\phi^i{}_{jkl} := (-1)^n \langle \star\hat{\phi}|_{T,e}(\tau^T_k, \tau^T_l)\tau^T_j, \tau^T_i\rangle_T = A|_{T,e}(\tau^T_k, \tau^T_l, \tau^T_i, \tau^T_j)$, so again $\hat{\phi}|_{T,e} = \frac{1}{2}\phi^i{}_{jkl}\tau^T_i \otimes \tau^{j^T} \otimes \star(\tau^{k^T} \wedge \tau^{l^T})$.

Here, because $\hat{\phi}|_T$ and $A|_T$ are both discontinuous on $\mathring{p}$, care needs to be taken about which of the “representatives” we use to evaluate $\phi^i{}_{jkl}$. We used the notation $\hat{\phi}|_{T,e}$ to refer to the value of $\hat{\phi}|_T$ that is continuously extended from $\mathring{e}$, i.e. $\hat{\phi}|_{T,e}(x) = \lim_{m\to\infty} \hat{\phi}|_T(x_m)$ where x_m is any sequence of points in $\mathring{e}$ that converge to $x \in p$, and likewise for A. This choice means that $i^*_{\mathring{p}}\hat{\phi}|_{T,e} = {\chi^T_{0,p}}^*\hat{\phi}^T$, although as discussed at the end of Subsection 2.2.5, the product we are evaluating ultimately does not depend on this choice.

Then we compute (abandoning Einstein notation):

$$\langle \hat{\Theta}^T_p \wedge {\chi^T_{0,p}}^* \hat{\phi}^T \rangle = \frac{1}{2}\Theta_p \sum_{i,j,k,l} \langle \tau^T_n \otimes \tau^{n-1^T} - \tau^T_{n-1} \otimes \tau^{n^T}, \phi^i{}_{jkl}\tau^T_i \otimes \tau^{j^T} \rangle i^*_{\mathring{p}} \star (\tau^{k^T} \wedge \tau^{l^T})$$

Note that, because $\tau^T_n = -\vec{n}^T$,

$$i^*_{\mathring{p}} \star (\tau^{k^T} \wedge \tau^{l^T}) = \begin{cases} -\omega_p & \text{if } k = n-1, l = n, \\ \omega_p & \text{if } k = n, l = n-1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\omega_p = -\tau^1 \wedge \cdots \wedge \tau^{n-2}$ is the volume form induced on p by the orientation of e. So this simplifies to

$$\begin{aligned}
&\frac{1}{2}\Theta_p \sum_{i,j,k,l} \langle \tau_n^T \otimes \tau^{n-1^T} - \tau_{n-1}^T \otimes \tau^{n^T}, {\phi^i}_{jkl} \tau_i^T \otimes \tau^{j^T} \rangle i_{\mathring{p}}^* \star (\tau^{k^T} \wedge \tau^{l^T}) \\
&\quad = \frac{1}{2}\Theta_p \sum_{i,j} \langle \tau_n^T \otimes \tau^{n-1^T} - \tau_{n-1}^T \otimes \tau^{n^T}, [{\phi^i}_{j,n,n-1} - {\phi^i}_{j,n-1,n}] \tau_i^T \otimes \tau^{j^T} \rangle \omega_p \\
&\quad = \Theta_p \sum_{i,j} \langle \tau_n \otimes \tau^{n-1^T} - \tau_{n-1} \otimes \tau^{n^T}, {\phi^i}_{j,n,n-1} \tau_i^T \otimes \tau^{j^T} \rangle \omega_p \\
&\quad = \Theta_p \sum_{i,j} (\delta_n^j \delta_i^{n-1} - \delta_{n-1}^j \delta_i^n) {\phi^i}_{j,n,n-1} \omega_p \\
&\quad = \Theta_p ({\phi^{n-1}}_{n,n,n-1} - {\phi^n}_{n-1,n,n-1}) \omega_p \\
&\quad = 2\Theta_p {\phi^{n-1}}_{n,n,n-1} \omega_p = 2\Theta_p A|_{T,e}(\tau_n^T, \tau_{n-1}^T, \tau_{n-1}^T, \tau_n^T) \omega_p = 2\Theta_p A_{\hat{\mu}\hat{\nu}\hat{\nu}\hat{\mu}} \omega_p.
\end{aligned}$$

□

Corollary 5. *Let $\hat{\phi} \in \mathcal{A}(f, \mathcal{T}, M)$ and let A be defined as in Proposition 1. Then, using notation from this paper on the left side and notation from [32] on the right side,*

$$\langle\langle \hat{R}_{\text{dist}}, \hat{\phi} \rangle\rangle = \frac{1}{2} \widetilde{\mathcal{R}\omega}(A)$$

Proof. The only term which is not straightforward from Proposition 1 and the definition of $\widetilde{\mathcal{R}\omega}$ in [32, p. 11] is the term involving the jump in second fundamental form across codimension-1 interfaces.

In $\hat{R}_{\text{dist}}$, the jump term is given by

$$-\int_{\mathring{e}} \langle \hat{\mathrm{I\!I}}_e^T \wedge i_{\mathring{e}}^* \hat{\phi}^T \rangle - \langle \hat{\mathrm{I\!I}}_e^{T'} \wedge i_{\mathring{e}}^* \hat{\phi}^{T'} \rangle,$$

where the orientation of the integral is chosen to be the same as the induced orientation on e from T. Per equation (5.2), this is the same as

$$2 \int_{\mathring{e}} \langle \mathrm{I\!I}^{\vec{\nu}^T}, A_{F\hat{\nu}\hat{\nu}F} \rangle \omega_e^T - \langle \mathrm{I\!I}^{\vec{\nu}^{T'}}, A_{F\hat{\nu}\hat{\nu}F} \rangle \omega_e^{T'}.$$

Next we note that $\omega_e^{T'} = -\omega_e^T$ because the two induced orientations are opposite, so this term simply becomes

$$2\int_{\mathring{e}} \langle \mathrm{I\!I}^{\vec{\nu}^T} + \mathrm{I\!I}^{\vec{\nu}^{T'}}, A_{F\hat{\nu}\hat{\nu}F} \rangle \omega_e^T = 2\int_{\mathring{e}} \langle [\![\mathrm{I\!I}]\!], A_{F\hat{\nu}\hat{\nu}F} \rangle \omega_e$$

Note that we can drop the superscript on ω_e^T because ω_e is implicitly the volume form induced by whatever orientation the integral is being evaluated with. $\square$

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